A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line

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Abstract. The Alesker–Poincaré pairing for smooth valuations on manifolds is expressed in terms of the Rumin differential operator acting on the cosphere-bundle. It is shown that the derivation operator, the signature operator and the Laplace operator acting on smooth valuations are formally self-adjoint with respect to this pairing. As an application, the product structure of the space of $SU(2)$- and translation invariant valuations on the quaternionic line is described. The principal kinematic formula on the quaternionic line $\mathbb{H}$ is stated and proved.

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1. Smooth valuations on manifolds

Let $M$ be a smooth manifold of dimension $n$. For simplicity, we suppose that $M$ is oriented, although the whole theory works in the non-oriented case as well. Following Alesker, we set $\mathcal{P}(M)$ to be the set of compact submanifolds with corners.

Definition 1.1. A valuation on $M$ is a real valued map $\mu$ on $\mathcal{P}(M)$ which is additive in the following sense: whenever $X, Y, X \cap Y$ and $X \cup Y$ belong to $\mathcal{P}(M)$, then

$$\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y).$$

A set $X \in \mathcal{P}(M)$ admits a conormal cycle $cnc(X)$, which is a compactly supported Legendrian cycle on the cosphere bundle $S^* M$. Sometimes it will be convenient to think of $S^* M$ as the set of pairs $(p, P)$ with $p \in M$ and $P \subset T_p M$ an oriented hyperplane, at other places it is better to think of it as the set of pairs $(p, [\xi])$ where $p \in M$ and $\xi \in T^*_p M \setminus \{0\}$ and the brackets denote the equivalence class for the relation $\xi_1 \sim \xi_2 \iff \xi_1 = \lambda \xi_2, \lambda > 0$.

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A valuation $\mu$ on $M$ is called smooth if there exist an $(n-1)$-form $\omega \in \Omega^{n-1}(S^*M)$ and an $n$-form $\phi \in \Omega^n(M)$ such that

$$\mu(X) = \cnc(X)(\omega) + \int_X \phi, \quad X \in \mathcal{P}(X).$$

(1)

If $\mu$ can be expressed in the form (1), we say that $\mu$ is represented by $(\omega, \phi)$. The space of smooth valuations on $M$ is denoted by $\mathcal{V}^\infty(M)$. It is a Fréchet space (see [6], Section 3.2 for the definition of the topology). If $M = V$ is a vector space, the subspace of translation invariant smooth valuations will be denoted by $\text{Val}^m(V)$.

Let $N$ be another oriented $n$-dimensional smooth manifold and $\rho: N \to M$ an orientation preserving immersion. Then $\rho$ induces a map $\tilde{\rho}: S^*N \to S^*M$, sending $(p, P)$ to $(\rho(p), T_{\rho(p)}P)$. It clearly satisfies $\pi \circ \tilde{\rho} = \rho \circ \pi$.

The valuation $\rho^*\mu$ on $N$ such that

$$\rho^*\mu(X) := \mu(\rho(X)), \quad X \in \mathcal{P}(N)$$

is again smooth. If $\mu$ is represented by $(\omega, \phi)$, then $\rho^*\mu$ is represented by $(\tilde{\rho}^*\omega, \rho^*\phi)$. This follows from the fact that $\cnc(\rho(X)) = \widetilde{\rho}_*\cnc(X)$. Note also that $\tilde{\rho}^{-1} = (\tilde{\rho})^{-1}$ if $\rho$ is a diffeomorphism.

We will use some results of [11], which we would like to recall. The cosphere bundle $S^*M$ is a contact manifold of dimension $2n-1$ with a global contact form $\alpha$ ($\alpha$ is not unique, but this will play no role here). The projection from $S^*M$ to $M$ will be denoted by $\pi$, it induces a linear map $\pi_*$ (fiber integration) on the level of forms.

Given an $(n-1)$-form $\omega$ on $S^*M$, there exists a unique vertical form $\alpha \wedge \xi$ such that $d(\omega + \alpha \wedge \xi)$ is vertical (i.e. a multiple of $\alpha$). The Rumin differential operator $D$ is defined as $D := d(\omega + \alpha \wedge \xi)$ [18]. The following theorem was proved in [11].

**Theorem 1.2.** Let $\omega \in \Omega^{n-1}(S^*M), \phi \in \Omega^n(M)$ and define the smooth valuation $\mu$ by (1). Then $\mu = 0$ if and only if

1. $D\omega + \pi^*\phi = 0$, and
2. $\pi_*\omega = 0$ for all $p \in M$.

Moreover, if $D\omega + \pi^*\phi = 0$, then $\mu$ is a multiple of the Euler characteristic $\chi$.

The support of a smooth valuation $\mu$ is defined as

$$\text{spt} \mu := M \setminus \{p \in M : \text{there exists } p \in U \subset M \text{ open such that } \mu|_U = 0\}.$$ 

The subspace of $\mathcal{V}^\infty(M)$ consisting of compactly supported valuations will be denoted by $\mathcal{V}^\infty_c(M)$.

Let $\int: \mathcal{V}^\infty_c(M) \to \mathbb{R}$ denote the integration functional [7]. If $\mu$ has compact support, then $\int \mu := \mu(X)$, where $X \in \mathcal{P}(M)$ is an $n$-dimensional manifold with
boundary containing $\text{spt} \mu$ in its interior. It is clear that, if $\mu$ is represented by $(\omega, \phi)$ with compact supports, then

$$\int \mu = \int_M \phi = [\phi] \in H^n_c(M) = \mathbb{R}. $$

Before stating our main theorem we have to recall two other constructions of Alesker.

The first one is the Euler–Verdier involution $\sigma : \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$ [6]. Let $s : S^*M \to S^*M$ be the natural involution on $S^*M$, sending $(p, P)$ to $(p, \bar{P})$, where $\bar{P}$ is the hyperplane $P$ with the reversed orientation. If a valuation $\mu \in \mathcal{V}^\infty(M)$ is represented by the pair $(\omega, \phi)$, then $\sigma \mu$ is defined as the valuation which is represented by the pair $((-1)^n s^* \omega, (-1)^n \phi)$.

The second construction is the Alesker–Fu product [9], which is a bilinear map

$$\mathcal{V}^\infty(M) \times \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M), \quad (\mu_1, \mu_2) \mapsto \mu_1 \cdot \mu_2.$$ We refer to [9] for its construction. It is characterized by the following properties:

1. “$\cdot$” is continuous and linear in both variables;
2. if $\rho : N \to M$ is a diffeomorphism and $\mu_1, \mu_2 \in \mathcal{V}^\infty(M)$, then
   $$\rho^* (\mu_1 \cdot \mu_2) = \rho^* \mu_1 \cdot \rho^* \mu_2;$$
3. if $m_1, m_2$ are smooth measures on an $n$-dimensional vector space $V$, $A_1, A_2 \in \mathcal{K}(V)$ convex bodies with strictly convex smooth boundary and if $\mu_i \in \mathcal{V}^\infty(V)$, $i = 1,2$ is defined by
   $$\mu_i(K) = m_i(K + A_i), \quad K \in \mathcal{K}(V), \quad (2)$$
then
   $$\mu_1 \cdot \mu_2(K) = m_1 \times m_2 (\Delta(K) + A_1 \times A_2),$$
where $\Delta : V \to V \times V$ is the diagonal embedding.

Our first main theorem is the following relation between Alesker–Fu product, integration functional, Euler–Verdier involution and Rumin differential.

**Theorem 1.3.** Let $\mu_1 \in \mathcal{V}^\infty(M)$ be represented by $(\omega_1, \phi_2)$; let $\mu_2 \in \mathcal{V}^\infty_c(M)$ be represented by $(\omega_2, \phi_2)$. Then

$$\int \mu_1 \cdot \sigma \mu_2 = (-1)^n \int_{S^*M} \omega_1 \wedge (D\omega_2 + \pi^* \phi_2) + \int_M \phi_1 \wedge \pi_* \omega_2. \quad (3)$$
Let us call the pairing

\[ \langle \mu_1, \mu_2 \rangle \mapsto \int_{M} \mu_1 \cdot \mu_2 =: \langle \mu_1, \mu_2 \rangle \]

the Alesker–Poincaré pairing. Note that Theorem 1.3 is equivalent to

\[ \langle \mu_1, \mu_2 \rangle = \int_{S^*M} \omega_1 \wedge s^*(D\omega_2 + \pi^*\phi_2) + \int_{M} \phi_1 \wedge \pi_*\omega_2. \]  

(5)

From Theorem 1.3 and from the fact that the Poincaré pairings on \( M \) and \( S^*M \) are perfect, we get the following corollary (which was first proved by Alesker).

**Corollary 1.4** ([7], Theorem 6.1.1). The Alesker–Poincaré pairing (4) is a perfect pairing.

Some more operators on \( V^\infty(M) \) were introduced in [11]. For this, we suppose that \( M \) is a Riemannian manifold. Then \( S^*M \) admits an induced metric, the Sasaki metric [20].

The first operator is the derivation operator \( \Lambda \) (which was denoted by \( \mathcal{L} \) in [11]). The metric on \( S^*M \) provides a canonical choice of \( \alpha \), namely \( \alpha(p, [\xi]) := \frac{1}{V_p} \pi^*\xi \) for all \((p, [\xi]) \in S^*M \). Let \( T \) be the Reeb vector field on \( S^*M \) (i.e. \( \alpha(T) = 1 \) and \( \mathcal{L}_T \alpha = 0 \)).

If the smooth valuation \( \mu \) is represented by \((\omega, \phi)\), then \( \Lambda \mu \) is by definition the valuation which is represented by \((\mathcal{L}_T\omega + i_T \pi^*\phi, 0)\).

Let us recall the definitions of the signature operator \( \mathcal{S} \) and the Laplace operator \( \Delta \). Let \( * \) be the Hodge star acting on \( \Omega^*(S^*M) \). Let \( \mu \in V^\infty(M) \) be represented by \((\omega, \phi)\). Then \( \mathcal{S} \mu \) is defined as the valuation which is represented by \((* (D\omega + \pi^*\phi), 0)\).

The Laplace operator \( \Delta \) is defined as \( \Delta := (-1)^n \mathcal{S}^2 \).

Our second main theorem shows that these operators fit well into Alesker’s theory. In fact, they are formally self-adjoint with respect to the Alesker–Poincaré pairing.

**Theorem 1.5.** For valuations \( \mu_1 \in V^\infty(M) \) and \( \mu_2 \in V^\infty_c(M) \), the following equations hold:

\[ \langle \Lambda \mu_1, \mu_2 \rangle = \langle \mu_1, \Lambda \mu_2 \rangle. \]  

(6)

\[ \langle \mathcal{S} \mu_1, \mu_2 \rangle = \langle \mu_1, \mathcal{S} \mu_2 \rangle. \]  

(7)

\[ \langle \Delta \mu_1, \mu_2 \rangle = \langle \mu_1, \Delta \mu_2 \rangle. \]  

(8)

We will apply these theorems in the study of the integral geometry of \( SU(2) \). This group acts on the quaternionic line \( \mathbb{H} \). In this setting, it is more natural to work with...
the space $K(\mathbb{H})$ of convex sets instead of manifolds with corners. By Proposition 2.6. of [8], there is no loss of generality in doing so.

It was shown by Alesker [3] that the space of SU(2)-invariant and translation invariant valuations on the quaternionic line $\mathbb{H}$ is of dimension 10. For each purely complex number $u$ of norm 1, let $I_u$ be the complex structure given by multiplication from the right with $u$ and $CP^1_u$ the corresponding Grassmannian of complex lines (with its unique SU(2)-invariant Haar measure). Alesker defined a valuation $Z_u$ by

$$Z_u(K) := \int_{CP^1_u} \text{vol}(\pi_L(K)) dL, \quad K \in K(\mathbb{H}).$$

He showed that $Z_i, Z_j, Z_k, Z_{i+j}, Z_{i+j}, Z_{i+j}$, together with Euler characteristic $\chi$, the volume $\text{vol}$ and the intrinsic volumes $\text{vol}_1, \text{vol}_3$ form a basis of $\text{Val}_{SU(2)}$. Following a suggestion of Fu, we will state the kinematic formula using a more symmetric choice. Noting that $Z_u = Z_{-u}$ for all $u \in S^2$, the 12 vertices $\pm u_i, i = 1, \ldots, 6$ of an icosahedron on $S^2$ define 6 valuations $Z_{u_i}, i = 1, \ldots, 6$.

We endow $SU(2)$ with its Haar measure and the semidirect product $SU(2) = SU(2) \ltimes \mathbb{H}$ with the product measure. Let $\text{vol}_k$ denote the $k$-dimensional intrinsic volume [15].

**Theorem 1.6** (Principal kinematic formula for $SU(2)$). Let $K, L \in K(\mathbb{H})$. Then

$$\int_{SU(2)} \chi(K \cap \tilde{g}L) d\tilde{g} = \chi(K) \text{vol}(L) + \frac{4}{3\pi} \text{vol}_1(K) \text{vol}_3(L)$$

$$+ \frac{17}{4} \sum_{i=1}^6 Z_{u_i}(K) Z_{u_i}(L) - \frac{3}{4} \sum_{1 \leq i < j \leq 6} Z_{u_i}(K) Z_{u_j}(L)$$

$$+ \frac{4}{3\pi} \text{vol}_3(K) \text{vol}_1(L) + \text{vol}(K) \chi(L).$$

This theorem implies and generalizes the Poincaré formulas of Tasaki [19], (which contained an error in some constant) as we will explain in the last section.

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**2. The Alesker–Poincaré pairing in terms of forms**

In order to prove Theorem 1.3 and Theorem 1.4, we will need three lemmas which are of independent interest.
Lemma 2.1 (Partition of unity for valuations, [7], Proposition 6.2.1). Let $M = \bigcup_i U_i$ be a locally finite open cover of $M$. Then there exist valuations $\mu_i \in \mathcal{V}^\infty(M)$ such that $\text{spt } \mu_i \subset U_i$ and

$$\sum_i \mu_i = \chi.$$ 

Proof. Let $1 = \sum_i f_i$ a partition of unity subordinate to $M = \bigcup_i U_i$. We represent $\chi$ by $(\omega, \phi)$ and let $\mu_i$ be the valuation represented by $(\pi^* f_i \wedge \omega, f_i \phi)$. \hfill \Box

By inspecting the proof of Theorem 1.2 (which uses a local variational argument), one gets the following lemma.

Lemma 2.2. Let $\omega \in \Omega^{n-1}(S^* M)$, $\phi \in \Omega^n(M)$ and define the smooth valuation $\mu$ by (1). Then

$$\text{spt } D\omega + \pi^* \phi \subset \pi^{-1}(\text{spt } \mu) \quad \text{and} \quad \text{spt } \pi_* \omega \subset \text{spt } \mu.$$ 

Lemma 2.3. Let $\mu \in \mathcal{V}^\infty(M)$ be compactly supported. Then $\mu$ can be represented by a pair $(\omega, \phi) \in \Omega^{n-1}(S^* M) \times \Omega^n(M)$ of compactly supported forms.

Proof. We suppose $M$ is non-compact (otherwise the statement is trivial). Let $\mu$ be represented by a pair $(\omega', \phi')$. Then $D\omega' + \pi^* \phi'$ is compactly supported. Since $H^n_c(S^* M) = \mathbb{R}$, there exists a compactly supported form $\phi \in \Omega^n_c(M)$ such that

$$[D\omega' + \pi^* \phi'] = [\pi^* \phi] \in H^n_c(S^* M).$$

In other words, there is a compactly supported form $\omega \in H^{n-1}_c(S^* M)$ such that $d\omega = D\omega = D\omega' + \pi^* \phi' - \pi^* \phi$. By Theorem 1.2, the pair $(\omega, \phi)$ represents $\mu$ up to a multiple of $\chi$. Since the valuation represented by $(\omega, \phi)$ and the valuation $\mu$ are both compactly supported, whereas $\chi$ is not, they have to be the same. \hfill \Box

Proof of Theorem 1.3. Note first that the right hand side of (3) is well defined: since $\mu_2$ is compactly supported, the same holds true for $D\omega_2 + \pi^* \phi_2$ and $\pi_* \omega_2$ by Lemma 2.2.

Next, both sides of (3) are linear in $\mu_1$ and $\mu_2$. Using Lemma 2.1, we may therefore assume that the supports of $\mu_1$ and $\mu_2$ are contained in the support of a coordinate chart. Since the Alesker–Fu product, the Euler–Verdier involution and the integration functional are natural with respect to diffeomorphisms, it suffices to prove (3) in the case where $M = V$ is a real vector space of dimension $n$.

Let us first suppose that $\mu_1$ and $\sigma \mu_2$ are of the type (2). We thus have $\mu_1(K) = m_1(K + A_1)$ and $\sigma \mu_2(K) = m_2(K + A_2)$ with smooth measures $m_1, m_2$ and smooth convex bodies $A_1, A_2$ with strictly convex boundary.
The left hand side of (3) is given by
\[
\int \mu_1 \cdot \sigma \mu_2 = m_1 \times m_2 (\Delta(V) + A_1 \times A_2) \\
= \int_V m_1((\Delta V + A_1 \times A_2) \cap V \times \{x\}) \, dm_2(x) \\
= \int_V m_1(x - A_2 + A_1) \, dm_2(x) \\
= \int_V \mu_1(x - A_2) \, dm_2(x) \\
= \int_V \cnc(x - A_2)(\omega_1) \, dm_2(x) + \int_V \left( \int_{x-A_2} \phi_1 \right) \, dm_2(x). \\
\tag{10}
\]

Let \( A \in \mathcal{K}(V) \) be smooth with strictly convex boundary. Its support function is defined by
\[
h_A : V^* \to \mathbb{R}, \\
h : \sup_{x \in A} \xi(x).
\]
Note that \( h_A \) is homogeneous of degree 1 and that \( h_{-A}(\xi) = h_A(-\xi). \)

Define the map \( G_A : S^* V \to S^* V, (x, [\xi]) \mapsto (x + d_{\xi} h_A, [\xi]) \) (since \( h_A \) is homogeneous of degree 1, \( d_{\xi} h_A \in V^{**} = V \) only depends on \( [\xi] \)). \( G_A \) is an orientation preserving diffeomorphism of \( S^* V \).

It is easy to show ([10], [12]) that for \( X \in \mathcal{K}(V) \)
\[
\cnc(X + A) = (G_A)_* \cnc(X). \\
\tag{11}
\]

We next compute that for all \( (x, [\xi]) \in S^* V \)
\[
G_A \circ s(x, [\xi]) = G_A(x, [-\xi]) = (x + d_{-\xi} h_{-A}, [-\xi]) = (x - d_{\xi} h_A, [\xi]) = s(x - d_{\xi} h_{-A}, [\xi]) = s \circ G_{-A}^{-1}(x, [\xi]). \\
\tag{12}
\]

Let \( \kappa_2 \in \Omega^n(V) \) be the form representing the measure \( m_2 \). The first term in (10) is equal to
\[
\int_V \cnc(x - A_2)(\omega_1) \, dm_2(x) = \int_V \cnc([x]) (G_{-A_2}^* \omega_1) \, dm_2(x) \\
= \int_V \pi_* (G_{-A_2}^* \omega_1) \wedge \kappa_2 \\
= \int_{S^* V} G_{-A_2}^* \omega_1 \wedge \pi^* \kappa_2 \\
= \int_{S^* V} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* \kappa_2. \\
\tag{13}
\]
By (11) we have $(-1)^n Ds^* \omega_2 + (-1)^n \pi^* \phi_2 = G^*_A \pi^* \kappa_2$. Applying $s^*$ to both sides and using (12), we get

$$(-1)^n (D \omega_2 + \pi^* \phi_2) = s^* G^*_A \pi^* \kappa_2 = (G^*_A)^{-1} \pi^* \kappa_2.$$  

Hence (13) equals $(-1)^n \int_{S^* V} \omega_1 \wedge (D \omega_2 + \pi^* \phi_2)$, which is the first term in (3).

By Fubini’s theorem, the second term in (10) equals

$$\int_V \left( \int_{S^2} \phi_1 \right) m_2(x) = \int_V m_2(y + A_2) \phi_1(y) = \int_V \sigma \mu_2(\{y\}) \phi_1(y).$$

For $y \in V$, we have $s_\ast \text{cnc}(\{y\}) = (-1)^n \text{cnc}(\{y\})$, since the antipodal map on $S^{n-1}$ is orientation preserving precisely if $n$ is even. Hence

$$\sigma \mu_2(\{y\}) = \pi_\ast \omega_2(y).$$

The second term in (10) thus equals $\int_V \phi_1 \wedge \pi_\ast \omega_2$, which corresponds to the second term in (3).

This finishes the proof in the case where $\mu_1$ and $\mu_2$ are of type (2). By linearity of both sides, (3) holds true for linear combinations of such valuations. Given arbitrary $\mu_1 \in \mathcal{V}^\infty(M)$ and $\mu_2 \in \mathcal{V}^\infty(M)$, we find sequences $\mu^j_1 \in \mathcal{V}^\infty(M)$ and $\mu^j_2 \in \mathcal{V}^\infty(M)$ such that $\mu^j_1 \to \mu_1$ and $\mu^j_2 \to \mu_2$ and such that $\mu^j_1$ and $\sigma \mu^j_2$ are linear combinations of valuations of type (2) (compare [5] and [6]).

By definition of the topology on $\mathcal{V}^\infty(M)$ (see Section 3.2 of [6]) and the open mapping theorem, there are sequences $(\omega^j_1, \phi^j_1)$ and $(\omega^j_2, \phi^j_2)$ representing $\mu^j_1, \mu^j_2$ and converging to $(\omega_1, \phi_1)$, $(\omega_2, \phi_2)$ in the $C^\infty$-topology. By what we have proved,

$$\int \mu^j_1 \cdot \sigma \mu^j_2 = (-1)^n \int_{S^* M} \omega^j_1 \wedge (D \omega^j_2 + \pi^* \phi^j_2) + \int_M \phi^j_1 \wedge \pi_\ast \omega^j_2$$

for all $j$. Letting $j$ tend to infinity, Equation (3) follows. \hfill $\square$

3. Self-adjointness of natural operators

Proof of Theorem 1.5. Note first the following equation:

$$\langle \sigma \mu_1, \mu_2 \rangle = (-1)^n (\mu_1, \sigma \mu_2).$$

(14)

This equation is immediate from (5) and the fact that $s \colon S^* M \to S^* M$ preserves orientation if and only if $n$ is even.

Let $\mu_1$ be represented by $(\omega_1, \phi_1)$. By Lemma 2.3 we may suppose that $\omega_2$ and $\phi_2$ are compactly supported.
\[ \Lambda \mu_i \text{ is represented by } \xi_i := i_T(D\omega_i + \pi^*\phi_i). \] Since \( D\omega_i + \pi^*\phi_i = \alpha \wedge \xi_i \), we get

\[
\langle \Lambda \mu_1, \sigma \mu_2 \rangle = (-1)^n \int_{S^*M} \xi_1 \wedge (D\omega_2 + \pi^*\phi_2)
\]
\[
= (-1)^n \int_{S^*M} \xi_1 \wedge \alpha \wedge \xi_2
\]
\[
= - \int_{S^*M} \xi_2 \wedge (D\omega_1 + \pi^*\phi_1)
\]
\[
= (-1)^{n+1} \int_{S^*M} \omega_1 \wedge D\xi_2 - \int_{M} \phi_1 \wedge \pi^*\xi_2
\]
\[
= - (\mu_1, \sigma \Lambda \mu_2).
\]  

(15)

Since \( D \) and \( s^* \) commute and since \( i_T \circ s^* = -s^* \circ i_T \), it is easily checked that

\[ \Lambda \circ \sigma = -\sigma \circ \Lambda. \]  

(16)

Now (6) follows from (15) and (16).

Let us next prove (7) ((8) is an immediate consequence).

By Lemma 2.3 we may suppose that \( \omega_2 \) and \( \phi_2 \) have compact support. Then

\[
\langle \mu_1, \sigma \delta \mu_2 \rangle = (-1)^n \int_{S^*M} \omega_1 \wedge D \ast (D\omega_2 + \pi^*\phi_2)
\]
\[
+ \int_{M} \phi_1 \wedge \pi^* \ast (D\omega_2 + \pi^*\phi_2)
\]
\[
= \int_{S^*M} (D\omega_1 + \pi^*\phi_1) \wedge \ast (D\omega_2 + \pi^*\phi_2)
\]
\[
= \int_{S^*M} \ast (D\omega_1 + \pi^*\phi_1) \wedge (D\omega_2 + \pi^*\phi_2)
\]
\[
= \int_{S^*M} (-1)^n s^* \ast (D\omega_1 + \pi^*\phi_1) \wedge s^* (D\omega_2 + \pi^*\phi_2)
\]
\[
= \langle \sigma \delta \mu_1, \mu_2 \rangle.
\]  

(17)

Since \( s \) changes the orientation of \( S^*M \) by \((-1)^n\), we get \( s^* \circ \ast = (-1)^n \ast \circ s^* \) on \( \Omega^*(S^*V) \). It follows that \( \sigma \circ \delta = (-1)^n \delta \circ \sigma \). Therefore (7) follows from (14) and (17).

Alesker defined the space \( \mathcal{V}^{-\infty}(M) \) of \textit{generalized valuations} on \( M \) by

\[ \mathcal{V}^{-\infty}(M) := \left( \mathcal{V}^\infty_c(M) \right)^*, \]

where the star means the topological dual. This space is endowed with the weak topology. By the perfectness of the Alesker–Poincaré pairing, there is a natural dense embedding \( \mathcal{V}^\infty_c(M) \hookrightarrow \mathcal{V}^{-\infty}(M) \).
**Corollary 3.1.** Let $M$ be a Riemannian manifold. Each of the operators $\Lambda$, $\Delta$ acting on $\mathcal{V}^\infty(M)$ admits a unique continuous extension to $\mathcal{V}^{-\infty}(M)$.

**Proof.** Uniqueness of the extension is clear, since $\mathcal{V}^\infty(M)$ is dense in $\mathcal{V}^{-\infty}(M)$. We let $\Lambda$ act on $\mathcal{V}^{-\infty}$ by $\Lambda\xi(\mu) := \xi(\Lambda\mu)$. By Theorem 1.5, this is consistent with the embedding of $\mathcal{V}^\infty(M)$ into $\mathcal{V}^{-\infty}(M)$ and we are done. The cases of $S$ and $\Delta$ are similar. \qed

4. The translation invariant case

From now on, $V$ will denote an oriented $n$-dimensional real vector space. We will consider valuations on the space $\mathcal{K}(V)$ of compact convex sets (i.e. convex valuations).

A convex valuation $\mu$ on $V$ is called translation invariant, if $\mu(x + K) = \mu(K)$ for all $K \in \mathcal{K}(V)$ and all $x \in V$.

A translation invariant convex valuation $\mu$ is said to be of degree $k$ if $\mu(tK) = t^k \mu(K)$ for $t > 0$ and $K \in \mathcal{K}(V)$. By $\text{Val}_k(V)$ we denote the space of translation invariant convex valuations of degree $k$. A valuation $\mu$ is even if $\mu(-K) = \mu(K)$ and odd if $\mu(-K) = -\mu(K)$, the corresponding spaces will be denoted by a superscript + or −.

In [16] it is shown that the space of translation invariant valuations can be written as a direct sum

$$\text{Val}(V) = \bigoplus_{k=0}^{n} \text{Val}_k(V).$$

Each space $\text{Val}_k(V)$ splits further as $\text{Val}_k(V) = \text{Val}_k^+(V) \oplus \text{Val}_k^-(V)$.

The spaces $\text{Val}_0(V)$ and $\text{Val}_n(V)$ are both 1-dimensional (generated by $\chi$ and a Lebesgue measure respectively). For $\mu \in \text{Val}(V)$, we denote by $\mu_n$ its component of degree $n$.

Let us prove the following version of Theorem 1.3 in the translation invariant case.

**Theorem 4.1.** Let $\mu_1, \mu_2 \in \text{Val}^n(V)$ be represented by translation invariant forms $(\omega_1, \phi_1)$, $(\omega_2, \phi_2)$ respectively. Then $(\mu_1 \cdot \sigma \mu_2)_n$ is represented by the $n$-form

$$(-1)^n \pi_* (\omega_1 \wedge (D \omega_2 + \pi^* \phi_2)) + \phi_1 \wedge \pi_* \omega_2 \in \Omega^n(V).$$

**Proof.** The proof is similar to that of Theorem 1.3. Fix a Euclidean metric on $V$. For $R > 0$, let $B_R$ denote the ball of radius $R$, centered at the origin. Let us suppose that
\( \mu_1(K) = \text{vol}(K + A_1) \) and \( \sigma \mu_2(K) = \text{vol}(K + A_2) \) for all \( K \in \mathcal{K}(V) \). Then

\[
\mu_1 \cdot \sigma \mu_2(B_R) = \text{vol}_2(\Delta(B_R) \times A_1 \times A_2)
\]

\[
= \int_{B_R} \text{vol}(x - A_2 + A_1) dx + o(R^n)
\]

\[
= \int_{B_R} \mu_1(x - A_2) + o(R^n)
\]

\[
= \int_{B_R} \text{cnc}(x - A_2)(\omega_1) dx + \int_{B_R} \int_{x - A_2} \phi_1 dx + o(R^n).
\]

The first term is given by

\[
\int_{B_R} \text{cnc}(x - A_2)(\omega_1) dx = \int_{B_R} \pi_*(G_{-A_2}^* \omega_1) dx + o(R^n)
\]

\[
= \int_{B_R \times S^*(V)} G_{-A_2}^* \omega_1 \wedge \pi^*(dx) + o(R^n)
\]

\[
= \int_{G_{-A_2}(B_R \times S^*(V))} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* dx + o(R^n)
\]

\[
= (-1)^n \int_{B_R \times S^*(V)} \omega_1 \wedge (D \omega_2 + \pi^* \phi_2) + o(R^n)
\]

\[
= (-1)^n \int_{B_R} \pi_*(\omega_1 \wedge (D \omega_2 + \pi^* \phi_2)) + o(R^n).
\]

The second term yields

\[
\int_{B_R} \int_{x - A_2} \phi_1 dx = \int_V \text{vol}((y + A_2) \cap B_R) \phi_1(y)
\]

\[
= \int_{B_R} \text{vol}(y + A_2) \phi_1(y) + o(R^n)
\]

\[
= \int_{B_R} \mu_2(\{y\}) \phi_1(y) + o(R^n)
\]

\[
= \int_{B_R} \phi_1 \wedge \pi^* \omega_2 + o(R^n).
\]

Therefore we obtain

\[
(\mu_1 \cdot \sigma \mu_2)_n = \lim_{R \to \infty} \frac{1}{R^n} \mu_1 \cdot \sigma \mu_2(B_R)
\]

\[
= \lim_{R \to \infty} \frac{1}{R^n} \int_{B_R} (-1)^n \pi_*(\omega_1 \wedge (D \omega_2 + \pi^* \phi_2)) + \phi_1 \wedge \pi^* \omega_2.
\]
This finishes the proof of Theorem 4.1 in the case where $\mu_1$ and $\sigma \mu_2$ are of type $K \mapsto \text{vol}(K + A)$. Using linearity of both sides, it also holds for linear combinations of such valuations. Since they are dense in $\text{Val}(V)$ (by Alesker’s solution of McMullen’s conjecture [1]), Theorem 4.1 is true in general.

Let us next suppose that $V$ is endowed with a Euclidean product. We can identify $\text{Val}_g(V)$ with $\mathbb{R}$ by sending $\text{vol}$ to 1. We get a symmetric bilinear form (called Alesker pairing)

\[
\text{Val}^{\text{sm}}(V) \times \text{Val}^{\text{sm}}(V) \to \mathbb{R},
\]

\[
(\mu_1, \mu_2) \mapsto \langle \mu_1, \mu_2 \rangle := (\mu_1 \cdot \mu_2)_n.
\]

**Corollary 4.2.** For valuations $\mu_1, \mu_2 \in \text{Val}^{\text{sm}}(V)$, the following equations hold:

\[
\langle \Lambda \mu_1, \mu_2 \rangle = \langle \mu_1, \Lambda \mu_2 \rangle,
\]

\[
\langle \delta \mu_1, \mu_2 \rangle = \langle \mu_1, \delta \mu_2 \rangle,
\]

\[
\langle \Delta \mu_1, \mu_2 \rangle = \langle \mu_1, \Delta \mu_2 \rangle.
\]

**Proof.** Analogous to the proof of Theorem 1.5. \qed

### 5. Kinematic formulas and Poincaré formulas

**5.1. Kinematic formulas.** In this section, we suppose that $G$ is a subgroup of $\text{O}(V)$ acting transitively on the unit sphere. By a result of Alesker [3], the space of translation invariant and $G$-invariant valuations $\text{Val}^G$ is a finite-dimensional vector space.

Let $\phi_1, \ldots, \phi_N$ a basis of $\text{Val}^G$. Suppose we have a kinematic formula

\[
\int_G \chi(K \cap \tilde{g} L) d \tilde{g} = \sum_{i,j=1}^N c_{i,j} \phi_i(K) \phi_j(L).
\]

Here and in the following, $G$ is endowed with its Haar measure and $\tilde{G} := G \ltimes V$ with the product measure.

Set

\[
k_G(\chi) := \sum_{i,j=1}^N c_{i,j} \phi_i \otimes \phi_j \in \text{Val}^G \otimes \text{Val}^G = \text{Hom}(\text{Val}^G, \text{Val}^{G*}).
\]

The Alesker pairing induces a bijective map

\[
\text{PD} \in \text{Hom}(\text{Val}^G, \text{Val}^{G*}).
\]
Fu [13] showed that these two maps are inverse to each other:

\[ k_G(\chi) = PD^{-1}. \] (18)

For further use, we give another interpretation of (18). Let \( G \) be as above. The scalar product on the finite-dimensional space \( \text{Val}^G \) induces a scalar product on \( \text{Val}^G \) such that \( PD \) is an isometry.

Given \( \mathcal{K} \subset \mathcal{K}(V) \), let \( \mu_K \in \text{Val}^G \) be defined by

\[ \mu_K(\mu) = \mu(K), \quad \mu \in \text{Val}^G. \]

**Proposition 5.1** (Principal kinematic formula). Let \( G \) be a subgroup of \( O(V) \) acting transitively on the unit sphere. Then for \( K, L \in \mathcal{K}(V) \)

\[ \int_G \chi(K \cap gL) \, d\bar{g} = \langle \mu_K, \mu_L \rangle. \]

**Proof.** Let \( \phi_1, \ldots, \phi_N \) be a basis of \( \text{Val}^G \). Set \( g_{ij} := \langle \phi_i, \phi_j \rangle \), \( i, j = 1, \ldots, N \). Let us denote by \( (g^{ij})_{i,j=1,\ldots,N} \) the inverse matrix. Then

\[ \int_G \chi(K \cap gL) \, d\bar{g} = \sum_{i,j} g^{ij} \mu_K(\phi_i) \mu_L(\phi_j) = \sum_{i,j} g^{ij} \mu_K(\phi_i) \mu_L(\phi_j) = \langle \mu_K, \mu_L \rangle. \] \( \square \)

### 5.2. Klain functions.

Let us suppose additionally that \(-1 \in G\), which implies that \( \text{Val}^G \subset \text{Val}^+ \).

For \( 0 \leq k \leq n \), the action of \( G \) on \( V \) induces an action on the Grassmannian \( \text{Gr}_k(V) \). We set \( \mathcal{P}_k := \text{Gr}_k(V)/G \) for the quotient space. Given \( u \in \mathcal{P}_k \), the space of \( k \)-planes contained in \( u \) admits a unique \( G \)-invariant probability measure and we define \( Z_u \in \text{Val}^G \) by

\[ Z_u(K) := \int_{L \in u} \text{vol}(\pi_L K) \, dL, \quad K \in \mathcal{K}(V). \]

Recall that the Klain function of an even, translation invariant valuation \( \mu \) of degree \( k \) on a Euclidean vector space \( V \) is the function \( \text{Kl}_\mu : \text{Gr}_k(V) \to \mathbb{R} \) such that the restriction of \( \mu \) to \( L \in \text{Gr}_k(V) \) is given by \( \text{Kl}_\mu(L) \) times the Lebesgue measure. An even, translation invariant valuation is uniquely determined by its Klain function [14]. If \( M \) is a compact \( k \)-dimensional submanifold (possibly with boundaries or corners), then

\[ \mu(M) = \int_M \text{Kl}_\mu(T_p M) \, dp. \]

Alesker proved the existence of a duality operator (or Fourier transform) \( \mathcal{F} \) on \( \text{Val}^{+,\text{sm}} \) such that \( \text{Kl}_{\mu} = \text{Kl}_{\mu \circ \mathcal{F}} \) for all \( \mu \in \text{Val}^{+,\text{sm}} \). \( \mathcal{F} \) is formally self-adjoint with respect to the Alesker pairing.
Proposition 5.2. Let $u, v \in P_k$ and $L \in v$. Then
\[ Kl(Z_u, Z_v). \] (19)

Proof. Immediate from Lemma 2.2. of [12].

Lemma 5.3. There are finitely many elements $u_1, \ldots, u_N$ such that $Z_{u_i}, i = 1, \ldots, N$ is a basis of $\text{Val}_k^G$.

Proof. Let $\phi_1, \ldots, \phi_N$ be a basis of $\text{Val}_k^G$. Let $m_i$ be the push-forward of a Crofton measure for $\phi_i$ on $G(V)$ under the projection $G(V) \rightarrow P_k$.

By $G$-invariance of $\phi_i$, we get
\[ \phi_i(K) = \int_{P_k} \int_{L \in u} \text{vol}(\pi_L K) dL d m_i(u). \]

Now, for sufficiently close approximations of the $m_i$ by discrete measures $\sum_{j=1}^k c_{i,j} \delta_{u_{i,j}}$ with $u_{i,j} \in P_k, c_{i,j} \in \mathbb{R}$, the valuations $\sum_j c_{i,j} Z_{u_{i,j}}$ form a basis of $\text{Val}_k^G$. Hence $\{Z_{u_{i,j}}, i = 1, \ldots, N, j = 1, \ldots, k\}$ is a finite generating set of $\text{Val}_k^G$, from which we can extract a finite basis.

5.3. Poincaré formulas. Poincaré formulas for $G$ are special cases of the principal kinematic formula for $G$, when $K$ and $L$ are replaced by smooth compact submanifolds $M_1$ and $M_2$ (possibly with boundary) of complementary dimension (note that $M_1, M_2 \in \mathcal{P}(V)$, so there is no problem in evaluating a valuation in $M_1$ and $M_2$). Then the right hand side of the principal kinematic formula is the “average number” of intersections of $M_1$ and $\bar{g} M_2$.

Proposition 5.4 (General Poincaré formula). Let $M_1, M_2$ be smooth compact submanifolds, possibly with boundaries, of complementary dimensions $k$ and $n-k$. Then
\[ \int_{\partial} \#(M_1 \cap \bar{g} M_2) d \bar{g} = \int_{M_1 \times M_2} \alpha(T_p M_1, T_q M_2) d p d q \]
with
\[ \alpha : P_k \times P_{n-k} \rightarrow \mathbb{R}, \]
\[ (u, v) \mapsto \langle Z_u, Z_v \rangle. \]

Proof. Let $u_1, \ldots, u_N$ be such that $Z_{u_i}, i = 1, \ldots, N$ is a basis of $\text{Val}_k^G$. Let $v_1, \ldots, v_N$ be such that $Z_{v_j}, j = 1, \ldots, N$ is a basis of $\text{Val}_{n-k}^G$ (note that the dimensions of these two spaces agree by Theorem 1.2.2 in [2]). Setting $g_{ij} := \langle Z_{u_i}, Z_{v_j} \rangle$
and \((g^{ij})\) for the inverse matrix, the principal kinematic formula implies that for all \(M_1\) and \(M_2\) as above
\[
\int_{\mathcal{G}} \#(M_1 \cap g M_2) \, dg = \sum_{i,j} g^{ij} Z_{u_i}(M_1) Z_{v_j}(M_2)
\]
\[
= \int_{M_1 \times M_2} \sum_{i,j} g^{ij} Kl_{Z_{u_i}}(T_p M_1) Kl_{Z_{v_j}}(T_q M_2) \, dp \, dq.
\]
This shows that
\[
\alpha(u, v) = \sum_{i,j} g^{ij} Kl_{Z_{u_i}}(u) Kl_{Z_{v_j}}(v) = (Z_u, Z_v);
\]
where the last equation follows from (19) and the self-adjointness of \(\mathbb{F}\).

6. Kinematic formulas for SU(2)

We apply the results of the preceding section to the special case \(G = SU(2)\) acting on the quaternionic line
\[
\mathbb{H} = \{ x_1 + x_2 i + x_3 j + x_4 k : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \}.
\]
Since this action is transitive on the unit sphere, \(\text{Val}^{SU(2)}\) is finite dimensional and \(\text{Val}^{SU(2)}_k\) is one-dimensional except for the case \(k = 2\). The quotient space \(P_2 := \text{Gr}_2(\mathbb{H})/SU(2)\) is the two-dimensional projective space \(\mathbb{R}P^2 = S^2/\{\pm 1\}\) ([19]). Following Tasaki, we denote by \((\omega_1(L) : \omega_2(L) : \omega_3(L)) \in \mathbb{R}P^2\) the class of \(L \in \text{Gr}_2(\mathbb{H})\).

A canonical representative in the preimage of \((a : b : c) \in \mathbb{R}P^2\) is given by the 2-plane spanned by 1 and \(ai + bj + ck\).

If \(u = (a, b, c) \in S^2\), then the planes in \(u\) are the complex lines for the complex structure \(I_u\) which is defined by multiplication by \(u\) from the right on \(\mathbb{H}\). We will therefore write \(\mathbb{C}P_u = \mathbb{C}P_{-u}\).

The following \(SU(2)\)-invariant and translation invariant valuations of degree 2 were introduced by Alesker [3].

**Definition 6.1.** Given \(u \in \mathbb{R}P^2\), set
\[
Z_u(K) := \int_{\mathbb{C}P_u} \text{vol}(\pi_L(K)) \, dL, \quad K \in \mathcal{K}(\mathbb{H}).
\]
Proof of Theorem 1.6. From Lemma 5.3 we infer that there is a finite number of points $u_1, \ldots, u_N \in \mathbb{R}P^2$ such that $Z_{u_i}, i = 1, \ldots, N$ form a basis of $\text{Val}_{2SU(2)}$. Alesker showed that $N = 6$ and that $Z_{i_1}, Z_{i_2}, Z_{i_3}, Z_{i_4}, Z_{i_5}, Z_{i_6}$ is such a basis.

Our aim is to compute the product $(Z_u, Z_v)$ for $u, v \in \mathbb{R}P^2$. We will achieve it by first expressing each $Z_u$ as a smooth valuation represented by some $3$-form $\omega_u \in \Omega^3(S^*\mathbb{H})$ and then applying Theorem 4.1.

Since the metric induces a diffeomorphism between $S^*\mathbb{H}$ and $S\mathbb{H}$, we may as well work with the latter space. The image of the conormal cycle of a compact convex set $K$ under this diffeomorphism is the normal cycle $nc(K)$.

Let us introduce several differential forms on $S\mathbb{H}$, depending on the choice of the complex structure $I_u$. We follow the notation of \[17\].

Let $\alpha, \beta, \gamma$ be $1$-forms on $S\mathbb{H}$ which, at a point $(x, v) \in S\mathbb{H}$, equal

\[
\alpha(w) = (v, d\pi(w)), \quad w \in T_{(x,v)}S\mathbb{H},
\]

\[
\beta_u(w) = (v, I_u d\pi(w)), \quad w \in T_{(x,v)}S\mathbb{H},
\]

\[
\gamma_u(w) = (v, I_u d\pi_2(w)), \quad w \in T_{(x,v)}S\mathbb{H}.
\]

Note that $\alpha$ is the canonical $1$-form (in particular independent of $u$), whereas $\beta_u$ and $\gamma_u$ depend on $u$.

Let $\Omega$ be the pull-back of the symplectic form on $(\mathbb{H}, I_u)$ to $S\mathbb{H}$, i.e.

\[
\Omega_u(w_1, w_2) := (d\pi(w_1), I_u d\pi(w_2)), \quad w_1, w_2 \in T_{(x,v)}S\mathbb{H}.
\]

Claim. $Z_u$ is represented by the $3$-form

\[
\omega_u := \frac{1}{8\pi} \beta_u \wedge d\beta_u + \frac{1}{4\pi} \gamma_u \wedge \Omega_u.
\]

Since $\omega_u$ is $U(2)$- and translation invariant and has bidegree $(2, 1)$ (with respect to the product decomposition $S\mathbb{H} = \mathbb{H} \times S(\mathbb{H})$), it represents some $U(2)$-invariant, translation invariant valuation $\mu_u$ of degree $2$. Here $U(2)$ is the unitary group for the complex structure $I_u$.

Now the space of valuations with these properties is of dimension $2$ \[2\]. It is thus enough to show that the valuation $Z_u$ and the valuation $\mu_u$ agree on the unit ball $B$ as well as on a complex disk $D_u$.

It is clear that $Z_u(B) = \omega_2 = \pi$. It was shown by Fu (compare Equation (37) in \[13\],) that $Z_u(D_u) = \frac{\pi}{2}$.

By \[11\], the derivation of a smooth translation invariant valuation $\mu$ on a finite-dimensional Euclidean vector space is given by

\[
\Lambda \mu(K) = \frac{d}{dt} \bigg|_{t=0} \mu(K + tB).
\]
It follows that, if \( \mu \) is of degree \( k \), then \( \Lambda \mu(B) = k \mu(B) \).

It is easily checked that \( \mathcal{L}_T \beta = \gamma, \mathcal{L}_T \gamma = 0 \) and \( \mathcal{L}_T^2 \Omega = d \gamma \), so that

\[
\mathcal{L}_T^2 \omega_u = \frac{1}{2\pi} \gamma \wedge d \gamma.
\]

Note that \( \gamma \wedge d \gamma \) is twice the volume form on \( S^3 \), hence \( \Lambda^2 \mu_u = 2\pi \chi \). It follows that

\[
\mu_u(B) = \frac{1}{2} \Lambda^2 \mu_u(B) = \pi.
\]

The restriction of \( \beta_u \) to the normal cycle of the complex disc \( D_u \) clearly vanishes. \( \gamma_u \) is the length element of the fibers of \( \pi : nc(D_u) \to D_u \) (which are circles), whereas \( \Omega_u \) is the (pull-back of) the volume form on \( D_u \). It follows that \( \omega_u(D_u) = \frac{\pi}{2} \).

The claim is proved.

Next, the Rumin operator is easily computed as

\[
\mathcal{D} \omega_u = d \left( \omega_u + \frac{1}{8\pi} \alpha \wedge \beta_u \wedge \gamma_u - \frac{1}{8\pi} \alpha \wedge \Omega_u \right) = \frac{1}{2\pi} \alpha \wedge \beta_u \wedge d \gamma_u.
\]

From Theorem 4.1 we infer that \( \mu_u \cdot \mu_v \) is represented by the 4-form

\[
\frac{1}{16\pi^2} \pi_*( (\beta_u \wedge d \beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d \gamma_v ) \in \Omega^4(\mathbb{H}).
\]

If \( u = (a : b : c) \) and \( v = (\tilde{a} : \tilde{b} : \tilde{c}) \), then

\[
(\beta_u \wedge d \beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d \gamma_v
\]

\[
= 2((\tilde{a} - a\tilde{b})^2 + (a\tilde{c} - \tilde{a}c)^2 + (b\tilde{c} - \tilde{b}c)^2 + 2(a\tilde{a} + b\tilde{b} + c\tilde{c})^2) d \text{vol}_{S^3} \]

\[
= 2(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2) d \text{vol}_{S^3}.
\]

It follows that

\[
\langle Z_u, Z_v \rangle = \frac{1}{4} (1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2).
\]  (20)

Let \( \pm u_i, i = 1, \ldots, 6 \) be the 12 vertices of an icosahedron \( I \) on \( S^2 \). They induce 6 valuations \( Z_{u_i}, i = 1, \ldots, 6 \). Since the edge length \( a \) of \( I \) satisfies \( \cos a = \frac{\sqrt{3}}{3} \), (20) implies that

\[
\langle Z_{u_i}, Z_{u_j} \rangle = \begin{cases} 
\frac{\pi}{3}, & i = j, \\
\frac{10}{3}, & i \neq j.
\end{cases}
\]  (21)

Theorem 1.6 follows easily from (21) and (18). \( \square \)
The general Poincaré formula (Proposition 5.4) implies the following (corrected version of the) Poincaré formula on the quaternionic line.

**Corollary 6.2** (Poincaré formula for SU(2), [19]). Let $M_1, M_2 \subset \mathbb{H}$ be compact smooth 2-dimensional submanifolds. Then

$$\int_{SU(2)} \#(M_1 \cap \tilde{g} M_2) d\tilde{g} = \frac{1}{4} \int_{M_1 \times M_2} (1 + A(T_p M_1, T_q M_2)) dp dq$$

with

$$A(T_p M_1, T_q M_2) = (\omega_1(T_p M_1)\omega_1(T_q M_2))$$

$$+ \omega_2(T_p M_1)\omega_2(T_q M_2) + \omega_3(T_p M_1)\omega_3(T_q M_2))^2.$$

**References**


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