Complete Determination of the Number of Galois Points for a Smooth Plane Curve

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Dedicated to my son Atsumu and my wife Kaori

Abstract - Let $C$ be a smooth plane curve. A point $P$ in the projective plane is said to be Galois with respect to $C$ if the function field extension induced by the projection from $P$ is Galois. We denote by $\delta(C)$ (resp. $\delta'(C)$) the number of Galois points contained in $C$ (resp. in $\mathbb{P}^2 \setminus C$). In this article, we determine the numbers $\delta(C)$ and $\delta'(C)$ in any remaining open cases. Summarizing results obtained by now, we will present a complete classification theorem of smooth plane curves by the number $\delta(C)$ or $\delta'(C)$. In particular, we give new characterizations of Fermat curve and Klein quartic curve by the number $\delta'(C)$.

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1. Introduction

Let the base field $K$ be an algebraically closed field of characteristic $p \geq 0$ and let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$. In 1996, H. Yoshihara introduced the notion of Galois point (see [14, 17] or survey paper [5]). If the function field extension $K(C)/K(\mathbb{P}^1)$, induced by the projection $\pi_P : C \to \mathbb{P}^1$ from a point $P \in \mathbb{P}^2$, is Galois, then the point $P$ is said to be Galois with respect to $C$. When a Galois point $P$ is contained in $C$ (resp. $\mathbb{P}^2 \setminus C$), we call $P$ an inner (resp. outer) Galois point. We denote by $\delta(C)$ (resp. $\delta'(C)$) the number of inner (resp. outer) Galois points for $C$. It is

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remarkable that many classification results of algebraic varieties have been given in the theory of Galois point.

Yoshihara and K. Miura determined $\delta(C)$ and $\delta'(C)$ in characteristic $p = 0$ ([14, 17]). In characteristic $p > 0$, M. Homma [13] settled $\delta(H)$ and $\delta'(H)$ for the Fermat curve $H$ of degree $p^e + 1$. Recently, the present author determined $\delta(C)$ when $p > 2$ or $d - 1$ is not a power of 2 ([3, 4]), and $\delta'(C)$ when $d$ is not divisible by $p$, $d = p$, or $d = 2^e$ in $p = 2$ ([3, 4, 7]). The following problems remain open ([4, Part III, Problem], [5, Problem 2]).

**Problem.** (1) Let $p = 2$ and let $e \geq 2$. Find and classify smooth plane curves of degree $d = 2^e + 1$ with $\delta(C) = d$.

(2) Let $p > 0$, $e \geq 1$ and let $d = p^e l$, where $l$ is not divisible by $p$. Assume that $(p^e, l) \neq (p, 1), (2^e, 1)$. Then, determine $\delta'(C)$.

In this article, we give a complete answer to these problems.

**Theorem 1.** Let $p = 2$, let $e \geq 2$ and let $C$ be a smooth plane curve of degree $d = 2^e + 1$. Then, $\delta(C) = d$ if and only if $C$ is projectively equivalent to the curve given by

\[(1) \quad \prod_{x \in \mathbb{F}_{2^e}} (x + xy + x^2) + cy^{2^e + 1} = 0,
\]

where $c \in K \setminus \{0, 1\}$.

**Theorem 2.** Let the characteristic $p > 0$, let $e \geq 1$, let $l$ be not divisible by $p$, and let $C$ be a smooth plane curve of degree $d = p^e l \geq 4$. If $(p^e, l) \neq (2^e, 1)$, then $\delta'(C) \leq 1$.

Summarizing Theorems 1 and 2 and the results of Yoshihara, Miura, Homma and the present author, we obtain the following classification theorem of smooth plane curves by the number $\delta(C)$ or $\delta'(C)$.

**Theorem 3** (Yoshihara, Miura, Homma, Fukasawa). Let $C$ be a smooth plane curve of degree $d \geq 4$ in characteristic $p \geq 0$. Then:

(I) $\delta(C) = 0, 1, d$ or $(d - 1)^3 + 1$. Furthermore, we have the following.

(i) $\delta(C) = (d - 1)^3 + 1$ if and only if $p > 0$, $d = p^e + 1$ and $C$ is projectively equivalent to the Fermat curve.

(ii) $\delta(C) = d \geq 5$ if and only if $p = 2$, $d = 2^e + 1$ and $C$ is projectively equivalent to the curve defined by

\[(2) \quad \prod_{x \in \mathbb{F}_{2^e}} (x + xy + x^2) + cy^{2^e + 1} = 0,
\]

where $c \in K \setminus \{0, 1\}$.
(iii) \(\delta(C) = d = 4\) if and only if \(p \neq 2, 3\) and \(C\) is projectively equivalent to the curve defined by \(x^3 + y^4 + 1 = 0\).

(II) \(\delta'(C) = 0, 1, 3, 7\) or \((d - 1)^4 - (d - 1)^3 + (d - 1)^2\). Furthermore, we have the following.

(i) \(\delta'(C) = (d - 1)^4 - (d - 1)^3 + (d - 1)^2\) if and only if \(p > 0, d - 1\) is a power of \(p\) and \(C\) is projectively equivalent to the Fermat curve.

(ii) \(\delta'(C) = 7\) if and only if \(p = 2, d = 4\) and \(C\) is projectively equivalent to Klein quartic curve.

(iii) \(\delta'(C) = 3\) and three Galois points are not contained in a common line if and only if \(d\) is not divisible by \(p\), \(d - 1\) is not a power of \(p\), and \(C\) is projectively equivalent to the Fermat curve.

(iv) \(\delta'(C) = 3\) and three Galois points are contained in a common line if and only if \(p = 2, d = 4\) and \(C\) is projectively equivalent to the curve defined by

\[(x^2 + x)^2 + (x^2 + x)(y^2 + y) + (y^2 + y)^2 + c = 0,

where \(c \in K \setminus \{0, 1\}\).

This is a modified and extended version of the paper [4, Part IV] (which will have been published only in arXiv).

2. Preliminaries

Let \(C \subset \mathbb{P}^2\) be a smooth plane curve of degree \(d \geq 4\) in characteristic \(p > 0\). For a point \(P \in C\), we denote by \(T_P C \subset \mathbb{P}^2\) the (projective) tangent line at \(P\). For a projective line \(l \subset \mathbb{P}^2\) and a point \(P \in C \cap l\), we denote by \(I_P(C, l)\) the intersection multiplicity of \(C\) and \(l\) at \(P\). We denote by \(\overline{PR}\) the line passing through points \(P\) and \(R\) when \(P \neq R\), and by \(\pi_P : C \to \mathbb{P}^1; R \mapsto \overline{PR}\) the projection from a point \(P \in \mathbb{P}^2\). If \(R \in C\), we denote by \(e_R\) the ramification index of \(\pi_P\) at \(R\). It is not difficult to check the following.

**Lemma 1.** Let \(P \in \mathbb{P}^2\) and let \(R \in C\). Then for \(\pi_P\) we have the following.

1. If \(R = P\), then \(e_R = I_R(C, T_R C) - 1\).
2. If \(R \neq P\), then \(e_R = I_R(C, \overline{PR})\).

Let \(P\) be a Galois point. We denote by \(G_P\) the group of birational maps from \(C\) to itself corresponding to the Galois group \(\text{Gal}(K(C)/\pi_P K(\mathbb{P}^1))\). We
find easily that the group $G_P$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(C)$ of $C$. We identify $G_P$ with the subgroup. When we use the symbol $\gamma$ for an automorphism of the curve $C$, we use the symbol $\gamma^*$ for the automorphism of the function field $K(C)$ corresponding to $\gamma$.

If a Galois covering $\theta : C \rightarrow C'$ between smooth curves is given, then the Galois group $G$ acts on $C$ naturally. We denote by $G(R)$ the stabilizer subgroup of $R$. The following fact is useful to find Galois points (see [15, III. 7.1, 7.2 and 8.2]).

**FACT 1.** Let $\theta : C \rightarrow C'$ be a Galois covering of degree $d$ with Galois group $G$ and let $R, R' \in C$. Then we have the following.

1. For any $\sigma \in G$, we have $\theta(\sigma(R)) = \theta(R)$.
2. If $\theta(R) = \theta(R')$, then there exists an element $\sigma \in G$ such that $\sigma(R) = R'$.
3. The order of $G(R)$ is equal to $e_R$ at $R$ for any point $R \in C$.
4. If $\theta(R) = \theta(R')$, then $e_R = e_{R'}$.
5. The index $e_R$ divides the degree $d$.

We recall a theorem on the structure of the Galois group at a Galois point (see [4, Part II]). Let $d - 1 = p^e l$ (resp. $d = p^e l$), where $l$ is not divisible by $p$, let $\zeta$ be a primitive $l$-th root of unity, and let $k = [\mathbb{F}_p(\zeta) : \mathbb{F}_p]$. Let $P = (1 : 0 : 0)$ be an inner (resp. outer) Galois point for $C$. The projection $\pi_P : C \rightarrow \mathbb{P}^1$ is given by $(x : y : 1) \mapsto (y : 1)$. We have a field extension $K(x, y)/K(y)$ via $\pi_P$. Let $\gamma \in G_P$. Then, the automorphism $\gamma \in G_P$ can be extended to a linear transformation of $\mathbb{P}^2$ (see [1, Appendix A, 17 and 18] or [2]). Let $A_\gamma = (a_{ij})$ be a $3 \times 3$ matrix representing $\gamma$. Since $\gamma \in G_P$, $\gamma^*(y) = y$. Then, $(a_{21} x + a_{22} y + a_{23}) - (a_{31} x + a_{32} y + a_{33}) y = 0$ in $K(x, y)$. Since $d \geq 4$, we have $a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22} = a_{33}$. We may assume that $a_{22} = a_{33} = 1$. Since $\gamma^{l e} = 1$, we have $a_{11} = 1$. We take a group homomorphism $G_P \rightarrow K \setminus 0; \gamma \mapsto a_{11}(\gamma)$, where $a_{11}(\gamma)$ is the $(1,1)$-element of $A_\gamma$. Then, we have the splitting exact sequence of groups

$$0 \rightarrow (\mathbb{Z}/p\mathbb{Z})^{\oplus e} \rightarrow G_P \rightarrow \langle \zeta \rangle \rightarrow 1,$$

and the following theorem.

**THEOREM 4.** Let $C \subset \mathbb{P}^2$ be a smooth curve and let $P$ be an inner (resp. outer) Galois point. Then, $k$ divides $e$ and $G_P \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus e} \times \langle \zeta \rangle$.

**REMARK 1.** The condition that $k$ divides $e$ is equivalent to that $l$ divides $p^e - 1$. We give a proof here. If $k$ divides $e$, $\mathbb{F}_p(\zeta) = \mathbb{F}_{p^k}$ is a subfield of $\mathbb{F}_{p^e}$. 
Since $\zeta \in \mathbb{F}_p^*$, $\zeta^{p'-1} = 1$. Since the order of $\zeta$ in the multiplicative group $\mathbb{F}_p^* \setminus 0$ is $l$, $l$ divides $p^e - 1$. The converse also holds.

We denote the kernel (resp. quotient) by $K_P$ (resp. by $Q_P$). An element $\sigma \in K_P$ (resp. a generator $\tau \in Q_P$) is represented by a matrix

$$A_\sigma = \begin{pmatrix} 1 & a_{12}(\sigma) & a_{13}(\sigma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{resp. } A_\tau = \begin{pmatrix} \zeta & a_{12}(\tau) & a_{13}(\tau) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a_{12}(\sigma), a_{13}(\sigma), a_{12}(\tau), a_{13}(\tau) \in K$. For each non-identity element $\gamma \in G_P$, there exist $\sigma \in K_P$ and $i$ such that $\gamma = \sigma^i$. Then, there exists a unique line $L_\gamma$, which is defined by $(\zeta^i - 1)X + (a_{12}(\sigma) + a_{12}(\tau^i))Y + (a_{13}(\sigma) + a_{13}(\tau^i))Z = 0$, such that $\gamma(R) = R$ for any $R \in L_\gamma$. Note that $P \in L_\gamma$ if and only if $\gamma \in K_P$. Furthermore, for $\sigma \in K_P$ and $R \neq P$, $L_\sigma = \overline{R\overline{P}}$ if and only if $\sigma(R) = R$. For a suitable system of coordinates, we can take $a_{12}(\tau) = a_{13}(\tau) = 0$.

Finally in this section, we note the following facts on automorphisms of $\mathbb{P}^1$.

**Lemma 2.** We denote by $\text{Aut}(\mathbb{P}^1)$ the automorphism group of $\mathbb{P}^1$.

1. Let $P_1, P_2, P_3 \in \mathbb{P}^1$ be three distinct points and let $\gamma_1, \gamma_2 \in \text{Aut}(\mathbb{P}^1)$. If $\gamma_1(P_i) = \gamma_2(P_i)$ for $i = 1, 2, 3$, then $\gamma_1 = \gamma_2$.
2. Let $P_1, P_2 \in \mathbb{P}^1$ be distinct points and let $G \subset \text{Aut}(\mathbb{P}^1)$ be a finite subgroup. If $\gamma(P_1) = P_1$ and $\gamma(P_2) = P_2$ for any $\gamma \in G$, then $G$ is a cyclic group whose order is not divisible by $p$ if $p > 0$.
3. Let $l$ be not divisible by $p$, let $P \in \mathbb{P}^1$, and let $G \subset \text{Aut}(\mathbb{P}^1)$ be a subgroup of order $l$. Assume that $G$ is cyclic and $\tau(P) = P$ for any $\tau \in G$. Then, there exists a unique point $Q$ such that $Q \neq P$ and $\tau(Q) = Q$ for any $\tau \in G$.

**Proof.** The fact (1) is easily proved, if we use the classical fact that any automorphism of $\mathbb{P}^1$ is a linear transformation. We prove (2). We may assume that $P_1 = (1 : 0)$ and $P_2 = (0 : 1)$. Let $\gamma \in G$. Since $\gamma(P_1) = P_1$ and $\gamma(P_2) = P_2$, $\gamma$ is represented by a matrix

$$A_\gamma = \begin{pmatrix} a(\gamma) & 0 \\ 0 & 1 \end{pmatrix},$$

where $a(\gamma) \in K$. Then, the homomorphism $\psi : G \rightarrow K \setminus 0 : \gamma \mapsto a(\gamma)$ is injective and $\psi(G)$ is cyclic. Let $m$ be the order of $\psi(G)$. Then, $\psi(G)$ is con-
tained in the set \( \{x \in K \mid 0 \mid x^m - 1 = 0 \} \). If \( m \) is divisible by \( p \), the set consists of at most \( m/p \) elements. Therefore, \( m \) is not divisible by \( p \). We have the conclusion.

We prove (3). We may assume that \( P = (1 : 0) \). Let \( \tau \) be a generator of \( G \). Since \( \tau(P) = P \) and \( \tau \) is an automorphism of order \( l \) not divisible by \( p \), \( \tau \) is represented by a matrix

\[
A_\tau = \begin{pmatrix} \zeta & b \\ 0 & 1 \end{pmatrix},
\]

where \( \zeta \) is a primitive \( l \)-th root of unity and \( b \in K \). Then, \( \tau^i \) is represented by the matrix

\[
A_{\tau^i} = \begin{pmatrix} \zeta^i & \zeta^i - 1 \\ 0 & 1 \end{pmatrix}.
\]

Let \( Q = (x : 1) \). Then, \( \tau^i(Q) = Q \) if and only if \( (\zeta - 1)x + b = 0 \). We have the conclusion. \( \Box \)

3. Only-if-part of the proof of Theorem 1

Let \( p = 2 \), let \( q = 2^e \geq 4 \) and let \( C \) be a plane curve of degree \( d = q + 1 \). Assume that \( \delta(C) = d \). Let \( P_1, \ldots, P_d \) be inner Galois points for \( C \). By the results of the previous paper [4, Part III, Lemma 1, Propositions 1, 3 and 4], we have the following.

**Proposition 1.** Assume that \( \delta(C) = d \). Then, we have the following.

1. Galois points \( P_1, \ldots, P_d \) are contained in a common line.
2. For any \( i \) and any element \( \sigma \in G_{P_i} \setminus \{1\} \), the order of \( \sigma \) is two.
3. For any \( i \) and any elements \( \sigma, \tau \in G_{P_i} \setminus \{1\} \) with \( \sigma \neq \tau \), \( L_\sigma \neq T_{P_i}C \) and \( L_\tau \neq L_\sigma \). In particular, the set \( \{T_{P_i}C \cap T_{P_j}C \mid 2 \leq i \leq d \} \) consists of exactly \( d - 1 \) points.

By the condition (1) and Fact 1(2), for each \( i \) with \( 3 \leq i \leq d \), there exists \( \tau_i \in G_{P_i} \) such that \( \tau_i(P_1) = P_2 \). Let \( \{Q\} = T_{P_1}C \cap T_{P_2}C \). In addition, we have the following by the condition (2).

4. For any \( i \) with \( 3 \leq i \leq d \), \( \tau_i(P_2) = P_1 \) and \( \tau_i(Q) = Q \).
5. For any \( i, j \) with \( 3 \leq i, j \leq d \), \( \tau_i \tau_j(P_1) = P_1 \), \( \tau_i \tau_j(P_2) = P_2 \) and \( \tau_i \tau_j(Q) = Q \).
Lemma 3. For a suitable system of coordinates, we may assume that $P_1 = (1:0:0)$, $P_2 = (0:0:1)$ and $Q = (0:1:0)$.

By Lemma 3 and Proposition 1(2)(4), $\tau_i$ is given by a matrix

$$A_{\tau_i} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a_i & 0 \\ a_i^2 & 0 & 0 \end{pmatrix},$$

for some $a_i \in K$. Then, $\tau_i \tau_j$ is given by the matrix

$$A_{\tau_i \tau_j} = \begin{pmatrix} a_j^2 & 0 & 0 \\ 0 & a_i a_j & 0 \\ 0 & 0 & a_i^2 \end{pmatrix}.$$

Let $H(C)$ be the subgroup of $\text{Aut}(\mathbb{P}^2)$ consisting of any $\gamma \in \text{Aut}(\mathbb{P}^2)$ satisfying

(h1) $\gamma(P_1) = P_1$, $\gamma(P_2) = P_2$ and $\gamma(Q) = Q$, 
(h2) $\{\gamma(P_i)|3 \leq i \leq d\} = \{P_i|3 \leq i \leq d\}$, and 
(h3) $\gamma(C) = C$.

Lemma 4. The group $H(C)$ is a cyclic group whose order is at most $d - 2 = q - 1$.

Proof. By the condition (h1) of $H(C)$, for any $\gamma \in H(C)$, $\gamma$ is represented by a matrix

$$A_{\gamma} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $a, b \in K$. We prove that $\gamma$ depends only on the image of $P_3$. Precisely, we show that for $\gamma_1, \gamma_2 \in H(C)$, if $\gamma_1(P_3) = \gamma_2(P_3)$, then $\gamma_1 = \gamma_2$. To prove this, it suffices to show that $\gamma = 1$ if $\gamma(P_3) = P_3$. Assume that $\gamma(P_3) = P_3$. Since $\gamma$ fixes three distinct points $P_1, P_2, P_3$ on the line $P_1P_2$, $\gamma$ is identity on the line $P_1P_2$, by Lemma 2(1) in Section 2. We have $a = 1$, since $P_1P_2$ is given by $Y = 0$. On the other hand, by the condition (h3), $\gamma(T_{P_3}C) = T_{P_3}C$. Then, the point $Q_0$ given by $T_{P_1}C \cap T_{P_3}C$ is fixed by $\gamma$. Note that $Q_0 \neq Q, P_1$ by Proposition 1(3)(1) and Fact 1(3). Since $\gamma$ fixes
three distinct points $P_1, Q, Q_0$ on the line $P_1Q$ and $P_1Q$ is given by $Z = 0$, we have $b = 1$.

By the above discussion and the condition (h2), the order of $H(C)$ is at most $d - 2 = q - 1$. We consider the group homomorphism $H(C) \to P_1P_2 \cong \mathbb{P}^1$ given by restrictions, which is well-defined by the condition (h1) of $H(C)$. Then, this is injective by the above discussion. It follows from Lemma 2(2) that $H(C)$ is cyclic. $\square$

We consider the set $S = \{\tau_3 \tau_i | 3 \leq i \leq d\}$. Then, $S \subset H(C)$ by Proposition 1(5)(1). Since the cardinality of $S$ is $q - 1$, $S = H(C)$ by Lemma 4. Since $H(C)$ is cyclic, there exists $i$ such that $\tau_3 \tau_i$ is a generator of $H(C)$. Therefore, $\tau_3 \tau_i$ is given by the matrix

$$A_{\tau_3 \tau_i} = \begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\zeta$ is a primitive $(q - 1)$-th root of unity. We denote $\tau_3 \tau_i$ by $\gamma$.

By Proposition 1(3), there exists an element $\sigma \in G_{P_1} \setminus \{1\}$ such that the $(1, 2)$-element $a_{12}(\sigma)$ and $(1, 3)$-element $a_{13}(\sigma)$ of a matrix $A_{\sigma}$ representing $\sigma$ are not zero (see also Section 2). If we take a linear transformation $\phi$ with $Y \mapsto (1/a_{12}(\sigma))Y$ and $Z \mapsto (1/a_{13}(\sigma))Z$, then $\phi(P_i) = P_i$ for $i = 1, 2, \phi(Q) = Q$, $\phi \circ \gamma \circ \phi^{-1} = \gamma$ and $\phi \circ \sigma \circ \phi^{-1}$ is represented by the matrix

$$A_0 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Therefore, we may assume that $\sigma$ is represented by the matrix $A_{\sigma} = A_0$. The automorphism $\gamma^j \sigma \gamma^{-j}$ is represented by the matrix

$$\begin{pmatrix} 1 & \zeta^j & \zeta^{2j} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

In particular, $\gamma^j \sigma \gamma^{-j} \in G_{P_1}$ for any $j$ with $1 \leq j \leq q - 1$. Since the cardinality of the set $\{\gamma^j \sigma \gamma^{-j} | 1 \leq j \leq q - 1\} \subset G_{P_1}$ is $q - 1$, $G_{P_1} = \{\gamma^j \sigma \gamma^{-j} | 1 \leq j \leq q - 1\} \cup \{1\}$. Then, the rational function $g(x, y) := \prod_{x \in \mathbb{F}_q} (x + xy + x^2) \in K(x, y)$ is fixed by any element of $G_{P_1}$. Therefore, $g(x, y) \in K(y)$. There exists $h(y) \in K(y)$ such that $g(x, y) + h(y) = 0$ in $K(x, y)$. Let $h(y) = h_1(y)/h_2(y)$, where $h_1, h_2 \in K[y]$. Then, $g(x, y)h_2(y) +
$h_1(y) = 0$ on $C$. Let $f(x, y)$ be a defining polynomial. Then, there exists $v(x, y) \in K[x, y]$ such that $f(x, y)v(x, y) = g(x, y)h_2(y) + h_1(y)$ as polynomials. Since $P_1 \subset C$ is smooth and the tangent line $T_{P_1}C = \overline{P_1Q}$ is given by $Z = 0$, the coefficient of $x^{2^c}$ for $f(x, y)$ as in $(K[y])[x]$ is a constant. Comparing the coefficient of degree $2^c$ in variable $x$, we have $v(x, y) \in K[y]$ and $v(x, y) = h_2(y)$ up to a constant. Then, $h_2(y)$ divides $h_1(y)$ and we have $h(y) \in K[y]$. Therefore, we may assume that $f(x, y) = g(x, y) + h(y)$, where $h(y) \in K[y]$. By the condition that the tangent line $T_{P_2}C = \overline{P_2Q}$ is given by $X = 0$, $g(x, y) + cy^{q+1} = 0$ for some $c \in K \setminus 0$. Therefore, we have a defining equation $f(x, y) = g(x, y) + cy^{q+1} = 0$.

Finally in this section, we investigate conditions for the smoothness of $C$. Let $G(X, Y, Z) := Z^{q+1}g(X/Z, Y/Z)$ and let $F(X, Y, Z) := Z^{q+1}f(X/Z, Y/Z)$. Then, by direct computations, we have $F(Z, Y, X) = F(X, Y, Z)$. Since there exist exactly $d$ points contained in $C$ and the line defined by $Y = 0$, such points are smooth. Therefore, singular points should lie on $Y \neq 0$. Let $h(x, z) = G(x, 1, z)$. We consider $h$ as an element of $K(z)[x]$. Then, the set \{ $z + x^2z \mid z \in F_q$ \}$ \subset K(z)$, which consists of all roots of $h(x, z) = 0$, forms an additive subgroup of $K(z)$. According to [8, Proposition 1.1.5 and Theorem 1.2.1], we have the following.

**Lemma 5.** The polynomial $h(x, z) \in K(z)[x]$ has only terms of degree equal to some power of $p$. In particular, $h_x(x, z) = z^q + z$, where $h_x$ is a partial derivative by $x$.

Assume that $(x, z) \in C$ is a singular point, i.e. $h_x(x, z) = h_z(x, z) = 0$. Then, $(x, z)$ is $F_q$-rational by Lemma 5. We have $c \neq 1$ by the following.

**Lemma 6.** The equality \{ $h(x, z) \mid x, z \in F_q$ \} = \{ 0, 1 \} holds.

**Proof.** If $z = 0$, then $h(x, z) = 0$. We fix $z_0 \in F_q \setminus 0$. We consider $h(x, z_0) = z_0 \prod_{x \in F_q} (x + \alpha + \alpha^2z_0) \in F_q[x]$. For each $x \in F_q$, there exists a unique $\beta \in F_q$ with $\beta \neq x$ such that $x + \alpha^2z_0 = \beta + \beta^2z_0$. Therefore, the cardinality of the set $S_0 := \{ x + \alpha^2z_0 \mid x, z \in F_q \}$ is $q/2 = 2^{q-1}$. By direct computations, we find that any element of $S_0$ is a root of the separable polynomial $h_0(x) = \sum_{i=0}^{q-1} z_0^{2^i}x^{2^i}$, which is of degree $q/2$. Then, $h(x, z_0) = h_0(x)^2$ as elements of $F_q[x]$. Then, by direct computations, we have $h_0(x)(h_0(x) + 1) = z_0(x^q + x)$.
as elements of \( \mathbb{F}_q[x] \). Assume \( x \in \mathbb{F}_q \). Then, \( h_0(x)(h_0(x) + 1) = 0 \). Therefore, 
\( h(x, z_0) = 0 \) or \( 1 \). If we take \( x \in \mathbb{F}_q \setminus S_0 \), then \( h_0(x) \neq 0 \) and hence, 
\( h(x, z_0) = 1 \).

\[ \square \]

4. If-part of the proof of Theorem 1

We use the same notation as in the previous section, \( g, f, F \), and so on. Let \( C \) be the plane curve given by Equation (1c) with \( c \in K \setminus \{0, 1\} \). As in the previous section, \( C \) is smooth. We prove \( \delta(C) = d \). We consider the projection \( \pi_{p_1} \) from \( P_1 = (1 : 0 : 0) \). Then, we have the field extension \( K(x, y)/K(y) \) with \( f(x, y) = g(x, y) + c y^{q+1} = 0 \). Since \((x + zy + x^2)^2 + \beta y + \beta^2 = x + (\alpha + \beta)y + (\alpha + \beta^2)^2\), we have \( f(x + zy + x^2, y) = f(x, y) \) for any \( x \in \mathbb{F}_q \). Therefore, \( P_1 \) is Galois. By the symmetric property of \( F(X, Y, Z) \) for \( X, Z \), we find that a point \((0 : 0 : 1)\) is also inner Galois for \( C \). We also find that there exist a tangent line \( T \) such that \( I_q(C, T) = 2 \) for some \( Q \in C \cap T \). Therefore, \( C \) is not projectively equivalent to the Fermat curve of degree \( q + 1 \) (see, for example, [13]). According to [4, Part III, Lemma 1 and Proposition 1], we have \( \delta(C) = d \).

Remark 2. The projective equivalence class of the plane curve given by Equation (1c) is uniquely determined by a constant \( c \in K \setminus 0 \). Therefore, infinitely many classes exist. Precisely, we have Lemma 7 below.

**Lemma 7.** Let \( a, b \in K \setminus 0 \) and let \( C_a \) (resp. \( C_b \)) be the plane curve given by Equation (1c) with \( c = a \) (resp. \( c = b \)). If there exists a projective transformation \( \phi \) such that \( \phi(C_a) = C_b \), then \( a = b \).

**Proof.** Let \( P_1, \ldots, P_d \) be inner Galois points for \( C_a \), which are contained in the line defined by \( Y = 0 \). Then, \( P_1, \ldots, P_d \) are also inner Galois for \( C_b \). Since the tangent lines \( T_i C_a \) and \( T_i C_b \) at \( P = (x^2 : 0 : 1) \) with \( \alpha \in \mathbb{F}_q \) are given by the same equation \( X + zY + x^2Z = 0 \), \( T_{P_i} C_a = T_{P_i} C_b \) for \( i = 1, \ldots, d \). We may assume that \( P_1 = (1 : 0 : 0), P_2 = (0 : 0 : 1) \) and \( P_3 = (1 : 1 : 1) \). Let \( Q_2 = (0 : 1 : 0) \) and let \( Q_3 = (1 : 1 : 0) \). Then, \( T_{P_1} C_a \cap T_{P_2} C_a = T_{P_1} C_b \cap T_{P_2} C_b = \{Q_2\} \) and \( T_{P_1} C_a \cap T_{P_2} C_a = T_{P_1} C_b \cap T_{P_2} C_b = \{Q_3\} \). Let \( \phi \) be a linear transformation such that \( \phi(C_a) = C_b \).

If \( \phi(P_1) = P_i \) for some \( i \neq 1 \), then we take \( \sigma \in G_{P_i}(C_b) \) for some \( j \) such that \( \sigma(P_i) = P_1 \), by Fact 1(2). Then, \( \sigma \circ \phi(P_1) = P_i \). Therefore, there exists a linear transformation \( \phi \) such that \( \phi(C_a) = C_b \) and \( \phi(P_1) = P_i \). If \( \phi(P_2) = P_i \) for some \( 3 \leq i \leq d \), then we take \( \tau \in G_{P_1}(C_b) \) such that
\( \tau(P_3) = P_2 \), by Fact 1(2). Then, \( \tau \circ \phi(P_2) = P_2 \). Therefore, there exists a linear transformation \( \phi \) such that \( \phi(C_a) = C_b \), \( \phi(P_1) = P_1 \) and \( \phi(P_2) = P_2 \). If \( \phi(P_3) = P_i \) for some \( 4 \leq i \leq d \), then we take \( \gamma \in H(C_b) \) such that \( \gamma(P_i) = P_3 \), where \( H(C_b) \) is the group for \( C_b \) discussed in the previous section. Then, \( \gamma \circ \phi(P_3) = P_3 \). Therefore, there exists a linear transformation \( \hat{\phi} \) such that \( \hat{\phi}(C_a) = C_b \), \( \hat{\phi}(P_i) = P_i \) for \( i = 1, 2, 3 \) and \( \hat{\phi}(Q_j) = Q_j \) for \( j = 2, 3 \). Since \( \hat{\phi}(P_1) = P_1, \hat{\phi}(P_2) = P_2 \) and \( \hat{\phi}(Q_2) = Q_2 \), \( \hat{\phi} \) is represented a matrix

\[
A_{\hat{\phi}} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]

for some \( \lambda_1, \lambda_2, \lambda_3 \in K \setminus 0 \). Since \( \hat{\phi}(P_3) = P_3 \) and \( \hat{\phi}(Q_3) = Q_3 \), we have \( \lambda_1 = \lambda_3 \) and \( \lambda_1 = \lambda_2 \). Then, \( \hat{\phi} \) is identity. Therefore, by considering the defining equations of \( C_a \) and \( C_b \), we should have \( a = b \). \( \square \)

**Remark 3.** If \( c = 1 \), then the plane curve \( C \) defined by Equation (1c) is parameterized as \( \mathbb{P}^1 \to \mathbb{P}^2 : (s : 1) \mapsto (s^{q+1} : s^q + s : 1) \). The distribution of Galois points for this curve has been settled in [6].

5. **Proof of Theorem 2 (The case where \( l \geq 3 \))**

If we have two outer Galois points, then we note the following (see Section 2).

**Lemma 8.** Let \( P, P_2 \) be outer Galois points for \( C \). Then, any element \( \gamma \in G_P \) can be extended to a linear transformation of \( \mathbb{P}^2 \), and hence \( \gamma(P_2) \in \mathbb{P}^2 \) is also outer Galois for \( C \).

Let \( d = p^e l \), where \( e \geq 1 \), \( l \geq 3 \) and \( l \) divides \( p^e - 1 \), and let \( P = (1 : 0 : 0) \) be an outer Galois point. It follows from a generalization of Pardini’s theorem by Hefez [9, (5.10) and (5.16)] and Homma [12] that the generic order of contact for \( C \) is equal to 2, i.e. \( I_R(C, T_R C) = 2 \) for a general point \( R \in C \) (see also [10, 11]).

Let \( M \subset \mathbb{P}^2 \) be a projective line with \( P \in M \). Note that \( \gamma(M) = M \) for any \( \gamma \in G_P \), by the forms of the matrices \( A_\gamma \) and \( A_\tau \) as in Section 2. The homomorphism \( r_P[M] : G_P \to \text{Aut}(M) \), which is induced by the restriction, is well-defined. Then, the kernel \( \text{Ker} \ r_P[M] \) is a subgroup of \( K_P \) and the cardinality of \( \text{Ker} \ r_P[M] \) is a power of \( p \), since \( \gamma \in \text{Ker} \ r_P[M] \) if and only if
\(L_\gamma = M\). We denote it by \(p^n[M]\). Since the kernel Ker \(r_P[M]\) is a subspace of \(G_P\) as \(\mathbb{F}_p\)-vector spaces, we have the following diagram.

\[
\begin{array}{ccc}
(Z/p\mathbb{Z})^{\oplus e[M]} & \cong & \text{Ker } r_P[M] \\
\downarrow & & \downarrow \\
0 & \rightarrow & (Z/p\mathbb{Z})^{\oplus e} \\
\downarrow & & \downarrow \\
0 & \rightarrow & (Z/p\mathbb{Z})^{\oplus e - v[M]} & \rightarrow & \text{Im } r_P[M] & \rightarrow & \langle \zeta \rangle & \rightarrow & 1
\end{array}
\]

Using lower splitting exact sequence as groups, we have the following.

**Lemma 9.** The integer \(l\) divides \(p^{e - v[M]} - 1\) for any line \(M\) with \(P \in M\).

Hereafter in this section, we assume that \(P_2 \in \mathbb{P}^2 \setminus \{P\}\) is an outer Galois point for \(C\).

**Proposition 2.** Assume that \(l \geq 3\). Then:

1. \(v[PP_2] = e\), and there exists a unique point \(Q \in \mathbb{P}^2\) with \(Q \neq P\) such that \(\gamma(Q) = Q\) for any \(\gamma \in G_P\).

Let \(Q\) be the point as in (1). Furthermore, we have the following.

2. If \(l \geq 5\), then \(P_2 = Q\).
3. If \(l = 4\) and \(P_2 \neq Q\), then \(Q \in C\) or there exist two outer Galois point \(P_3, P_4\) such that \(\gamma(P_4) = P_4\) for any \(\gamma \in G_P\).
4. If \(l = 3\) and \(P_2 \neq Q\), then \(Q \in C\).

**Proof.** Let \(\gamma \in G_P \setminus \mathcal{K}_P\) and let \(L_\gamma\) be the line, which is a fixed locus, defined as in Section 2. The set \(C \cap L_\gamma\) consists of \(d\) points, because \(T_R C = \overline{PR} \neq L\), if \(R \in C \cap L_\gamma\) by Fact 1(3) and Lemma 1(2). Let \(P_2 \in Q_P\) be a generator and let \(L_\tau\) be the line, defined as in Section 2. We denote \(v[PP_2]\) by \(v\) and assume that \(v < e\).

We consider the case where \(\gamma(P_2) = P_2\) for some \(\gamma \in G_P \setminus \mathcal{K}_P\). Let \(\sigma \in \mathcal{K}_P\). Then, \(\sigma(R) \in L_\gamma\) and \(\sigma(R) \neq R\) if \(R \in C \cap L_\gamma\), by Fact 1(1)(3) and that \(L_\gamma\) consists of exactly \(d\) points. Furthermore, \(\sigma(P) \in T_{\sigma(R)} C \cap T_{\sigma(R_2)} C = \{P\}\), if \(R_1, R_2 \in C \cap L_\gamma\) with \(R_1 \neq R_2\). Therefore, we should have \(\sigma(P) = P\). This is a contradiction to \(v < e\).

We consider the case where \(\gamma(P_2) \neq P_2\) for any \(\gamma \in G_P \setminus \mathcal{K}_P\). Assume that \(\gamma_1(P_2) = \gamma_2(P_2)\) for \(\gamma_1, \gamma_2 \in G_P\). Note that any element \(\gamma \in G_P\) is represented as \(\gamma = \sigma \tau^i\) for some \(\sigma \in \mathcal{K}_P\) and some \(i\) (see Section 2). Let
\[ \gamma_1 = \sigma_1^i \] \text{and} \[ \gamma_2 = \sigma_2^j, \] where \( \sigma_1, \sigma_2 \in \mathbb{K}_p. \) Since \((\tau^{-i}\sigma_2^{-1}\sigma_1^i \gamma_2)\tau^{-i-j}(P_2) = P_2 \) and \( \tau^{-j}\sigma_2^{-1}\sigma_1^i \gamma_2 \in \mathbb{K}_p, \) we have \( i = j \) and \( \gamma_2^{-1}\gamma_1 \in \mathbb{K}_p \) by the assumption. Furthermore, we have \( \gamma_2^{-1}\gamma_1 \in \ker r_P[PP_2], \) since \( \gamma_2^{-1}\gamma_1(P) = P \) and \( \gamma_2^{-1}\gamma_1(P_2) = P_2. \) Therefore, we have \( p^{e-v}l + 1 \) outer Galois points on the line \( PP_2 \), by Lemma 8 and that the group \( \text{Im} r_P[PP_2] \) is isomorphic to \((\mathbb{Z}/p\mathbb{Z})^{e-v} \times \langle \zeta \rangle. \)

Let \( R \in C \cap L_i. \) We consider points on the line \( PP \). Let \( \overline{G_P}(R) = p^b l, \) where \( G_P(R) \) is the stabilizer subgroup at \( R. \) Then we have \( p^{e-b} \) flexes of order \( (\overline{G_P}(R) - 2) \) by Fact 1(3). We note that \( (p^b l - 2)p^{e-b} \geq p^e(l - 2). \) Furthermore, for each outer Galois points, we spent at least degree \((d - 1)(p^e(l - 2))\) as the degree of the Wronskian divisor. Therefore, it follows from the degree of Wronskian divisor ([16, Theorem 1.5]) that

\[ (p^{e-v}l + 1)(d - 1)p^e(l - 2) \leq 3d(d - 2). \]

Then, we have

\[ (p^{e-v}l + 1)p^e(l - 2) < 3d = 3p^e l. \]

Therefore, \( (p^{e-v}l + 1)(l - 2) - 3l < 0. \) Note that \( p^{e-v} - 1 \geq l \) by Lemma 9. Then, \( l^2 + l + 1)(l - 2) - 3l < 0. \) This is a contradiction. Therefore, \( v = e. \)

In particular, the group \( \text{Im} r_P[PP_2] \) is a cyclic group of order \( l. \) By Lemma 2(3) in Section 2, a fixed point by the group \( \text{Im} r_P[PP_2] \) which is different from \( P \) is uniquely determined. We denote it by \( Q. \) Then, \( \gamma(Q) = Q \) for any \( \gamma \in G_P, \) since \( \gamma = \sigma^i \) for some \( \sigma \in \mathbb{K}_P \) and some \( i. \) We have (1).

We prove (2). Assume that \( P_2 \neq Q. \) Since the group \( \text{Im} r_P[PP_2] \) is a cyclic group of order \( l, \) we have \( l + 1 \) outer Galois points on the line \( PP_2, \) by Lemma 8. Furthermore, for each outer Galois point, we spent at least degree \((d - 1)p^e(l - 2)\) as the degree of the Wronskian divisor, similarly to the proof of (1). Therefore, it follows from the degree of Wronskian divisor ([16, Theorem 1.5]) that

\[ (l + 1)(d - 1)p^e(l - 2) \leq 3d(d - 2). \]

Then, we have

\[ (l + 1)p^e(l - 2) < 3d = 3p^e l. \]

Therefore, \( (l + 1)(l - 2) - 3l < 0. \) Then, \( l^2 - 4l - 2 < 0. \) We have \( l \leq 4. \)

We prove (3). Assume that \( P_2 \neq Q \) and \( Q \notin C. \) Since \( \text{Im} r_P[PP_2] \) is a cyclic group of order \( l, \) the cardinality of \( C \cap PP_2 \) is equal to \( l \) and there exists \( l + 1 \) outer Galois points on \( PP_2, \) by Fact 1(3), Lemma 8 and the assumption. Let \( C \cap PP_2 = \{ R_1, \ldots, R_l \} \) and let \( P, P_2, \ldots, P_{l+1} \) be outer Galois points.
Let $l = 4$. The restriction $r_P[PP_2](r)$ of the generator $r \in Q_P$ is a generator of $\text{Im} \ r_P[PP_2]$. We may assume that $r_i = R_{i+1}$ for $i = 1, 2, 3, 4$, where $R_2 = R_1$. We can take $\eta_i \in \text{Im} \ r_P[PP_2]$ such that $\eta_i(R_1) = R_2$ for $j = 2, 3, 4, 5$ by Fact 1(2). We consider the case where at least three elements of $\{\eta_i\}$ are of order 4. We may assume that $\eta_2, \eta_3, \eta_4$ are of order 4. Assume that $\eta_j(R_2) = R_4$ for any $j$ with $2 \leq j \leq 4$. Then, we have $\eta_j(R_4) = R_3$. Since three points on the line $PP_2$ have the same images under $\eta_2, \eta_3, \eta_4$, these are the same automorphism of the line $PP_2$ by Lemma 2(1). Then, $\eta_j$ fixes $P_2, P_3, P_4$ for $j = 2, 3, 4$, because $\eta_j(P_j) = P_j$. This implies that $\eta_j$ is identity on $PP_2$, by Lemma 2(1). This is a contradiction. Therefore, there exists $j$ such that $\eta_j(R_2) = R_3$. Then, we have $\eta_j(R_3) = R_4$. Therefore, $\tau$ coincides with $\eta_j$ on the line $PP_2$. Then, $\tau(P_j) = \eta_j(P_j) = P_j \neq Q$. This implies that $\tau$ fixes $P_1, P_j$ and $Q$. This is a contradiction.

We consider the case where there exist distinct $j, k$ such that $\eta_j$ and $\eta_k$ is of order 2. Then, $\eta_j(R_2) = R_1, \eta_j(R_3) = R_4$ and $\eta_j(R_4) = R_3$. This holds also for $\eta_k$. Then $\eta_i = \eta_k$ on the line $PP_2$ by Lemma 2(1). Then, $\eta_j(P_k) = \eta_k(P_k) = P_k$. Since the group $\text{Im} \ r_P[PP_2]$ is cyclic, $\eta(P_k) = P_k$ for any $\eta \in \text{Im} \ r_P[PP_2]$, by Lemma 2(3). If we take $j = 3$ and $k = 4$, then we have the conclusion, since any element of $G_P$ is a product of elements of $K_P$ and of $Q_P$.

We prove (4). Let $l = 3$. Assume that $P_2 \neq Q$ and $Q \notin C$. We may assume that $\tau \in G_P$ satisfies that $\tau(R_i) = R_{i+1}$ for $i = 1, 2, 3$, where $R_4 = R_1$. We can take $\eta \in G_{PP_2}$ such that $\eta(R_1) = R_2$, by Fact 1(2). Then, we have $\eta(R_2) = R_3$ and $\eta(R_3) = R_1$. This implies that $\tau$ coincides with $\eta$ on $PP_2$, by Lemma 2(1). Therefore, $\tau(P_2) = \eta(P_2) = P_2 \neq Q$. This is a contradiction. □

Let $Q \in \mathbb{P}^2 \setminus \{P\}$ be the point such that $\gamma(Q) = Q$ for any $\gamma \in G_P$, as in Proposition 2. We may assume that $Q = (0 : 1 : 0)$ for a suitable system of coordinates. Then, the line $\overline{PQ} = PP_2$ is defined by $Z = 0$. Using Proposition 2(1), we can determine the defining equation of $C$, as follows.

**Proposition 3.** The curve $C$ is projectively equivalent to a plane curve whose defining equation is of the form $f(x, y) = \left( \sum_{0 \leq m \leq l} a_m x^m y^{l-m} \right)^l + h(y) = 0$, where $a_c, \ldots, a_0 \in K$ and $h(y) \in K[y]$ is a polynomial. Furthermore, $a_c x_0 \neq 0$, the derivative $h'(y)$ is of degree $d - 2$, and polynomials $h(y)$ and $h'(y)$ do not have a common root.

**Proof.** Let $\sigma \in K_P$ and let $r \in Q_P$ be a generator, as in Section 2. We may assume that $\tau^r(x) = \zeta x$ and $\tau^r y = y$ for $\tau^r : K(C) \to K(C)$, where $\zeta$ is a
primitive \( l \)-th root of unity. Let \( A_\sigma \) be a matrix representing \( \sigma \in \mathcal{K}_P \) as in Section 2. Since \( L_\sigma \) is defined by \( Z = 0 \), the \((1,2)\)-element of \( A_\sigma \) is zero. Since the group \( \mathcal{K}_P \) is a \( \mathbb{F}_p(\zeta) \)-vector space, we have a system of basis \( b_1, \ldots, b_m \), where \( km = e \). For any \( \sigma \in \mathcal{K}_P \), the \((1,3)\)-element of \( A_\sigma \) is given by \( z_1 b_1 + \cdots + z_m b_m \) for some \( (z_1, \ldots, z_m) \in \mathbb{F}_p(\zeta)^m \). We define \( g_0(x) = \prod_{(z_1, \ldots, z_m)} (x + \sum_i z_i b_i) \), where the subscript \( (z_1, \ldots, z_m) \in \mathbb{F}_p(\zeta)^m \) is taken over all elements. Let \( g = g_0^\ell \). Then, we find easily that \( \gamma g(x) = g(x) \) for any element \( \gamma \in G_P \). Therefore, there exists an element \( h(y) \in K(y) \) such that \( g(x) + h(y) = 0 \) in \( K(C) \). Then, \( h(y) \) is a polynomial of degree at most \( d \) by considering the degree of \( C \). On the other hand, the set \( \left\{ \sum_i z_i b_i | z_i \in \mathbb{F}_p(\zeta) \right\} \subset K \), which consists of all roots of \( g_0(x) = 0 \), forms an additive subgroup of \( K \). According to [8, Proposition 1.1.5 and Theorem 1.2.1], the polynomial \( g_0 \) has only terms of degree equal to some power of \( p \), i.e. \( g_0 = z_0 x^{p^\ell} + \cdots + z_1 x + z_0 \) for some \( z_0, \ldots, z_0 \in K \). Since \( g_0 \) is separable and has \( p^e \) roots, we have \( z_e z_0 \neq 0 \).

Finally, we prove that the degree of \( h'(y) \) is \( d - 2 \), and \( h(y) \) and \( h'(y) \) do not have a common root. Since \( h(y) \) is of degree at most \( d = p^\ell \), \( h'(y) \) is of degree at most \( d - 2 \). Let \( F(X, Y, Z) = f(X/Z, Y/Z)Z^d \), \( G_0(X, Z) = g_0(X/Z)Z^{p^\ell} \) and \( H(Y, Z) = h(Y/Z)Z^d \). Then, \( F_X = lG_0^{l-1}(\alpha_0 Z^{p^\ell-1}), F_Y = H_Y \) and \( F_Z = lG_0^{l-1}(\alpha_0 Z^{p^\ell-2}) + H_Z \). We have \( F_X(X, Y, 0) = 0 \). Since \( d = p^\ell \), \( F_Y(X, Y, 0) = H_Y(Y, 0) = 0 \). Assume that \( h'(y) \) is of degree at most \( d - 3 \). Then, \( F_Z(X, Y, 0) = 0 + H_Z(Y, 0) = 0 \). Therefore, \( C \) has singular points on the line defined by \( Z = 0 \). This is a contradiction to the smoothness of \( C \). On the other hand, if there exist \( b \in K \) such that \( h(b) = h'(b) = 0 \), then a point \((a : b : 1)\) with \( g_0(a) = 0 \) is a singular point.

**Lemma 10.** Let \( C \) be the plane curve given by the equation as in Proposition 3. Then, \( Q \in \mathbb{P}^2 \setminus C \) and \( Q \not\in P_2 \).

**Proof.** It follows from Lemma 2(1) and Fact 1(4) that \( L_\sigma = \overline{PP_2} \) for any \( \sigma \in \mathcal{K}_P \). Therefore, any ramified point \( R \in C \) of \( \pi_2 \) with \( Z \neq 0 \) is tame. Let \( \pi_Q \) be the projection from \( Q \). Note that \( \pi_Q(x : y : 1) = (x : 1) \). By the form of \( \pi_Q \), if \( x - x_0 \) is a local parameter at \((x_0, y_0) \in C \), then \((x_0, y_0) \) is not a ramification point. For a point \((x_0, y_0) \) with \( f_\Sigma(x_0, y_0) = l g_0(x_0)^{l-1} \neq 0, y - y_0 \) is a local parameter. Therefore, ramification points of \( \pi_Q \) in \( Z \neq 0 \) are contained in the locus \( \frac{dx}{dy} = -\frac{h'(y)}{f_\Sigma} = 0 \), which is equivalent to \( h'(y) = 0 \). Therefore, there exist \( d - 2 \) lines \( l_1, \ldots, l_{d-2} \) which contain \( P \) and \( d \) rami-
fication points of \( \pi_Q \), by Proposition 3. Since \( P_2 \neq P \), for any ramification point \( R \) of \( \pi_Q \), the cardinality of the set \( \overline{P_2R} \cap \{ R' \in C | Q \in T_R C \} \subset \overline{P_2R} \cap \bigcup_{i=1}^{d-2} l_i \) is at most \( d - 2 \).

Assume that \( Q \in C \). It follows from Fact 1(3) that \( I_Q(C, \overline{PQ}) = d \). By Fact 1(3) again, \( \gamma(Q) = Q \) for any \( \gamma \in G_{P_2} \). Let \( R \in C \) be a ramification point of \( \pi_Q \) in \( Z \neq 0 \). It follows from Lemma 1(2) that \( Q \in T_R C \). Since \( \gamma(Q) = Q \) for any \( \gamma \in G_{P_2} \), \( Q \in T_{\gamma(R)} C \) for any \( \gamma \in G_{P_2} \). Then, the cardinality of \( C \cap \overline{P_2R} \) is \( d \) and \( Q \in T_R C \) for any \( R' \in C \cap \overline{P_2R} \). This is a contradiction to the fact that the cardinality of \( \overline{P_2R} \cap \{ R' \in C | Q \in T_R C \} \) is at most \( d - 2 \). Therefore, \( Q \in P^2 \setminus C \).

Assume that \( Q \in P_2 \) and \( Q = P_2 \). Then, the set \( C \cap \overline{P_2P_2} \) contains \( l \) points, since the group \( \text{Im} r_P[\overline{P_2P_2}] \) is cyclic of order \( l \). Let \( \tau_2 \in \mathcal{Q}_{P_2} \) be a generator and let \( L_{\tau_2} \) be the line defined as in Section 2. Then, the locus \( \Sigma = \bigcup_{\sigma \in K_{P_2}} \sigma(L_{\tau_2}) \) consists of \( p^e \) lines. By considering the order of \( G_{P_2} \), the ramification locus of \( \pi_Q \) in the affine plane \( Z \neq 0 \) is contained in the locus \( \Sigma \).

Note that the set \( \bigcap_{\sigma \in K_{P_2}} \sigma(L_{\tau_2}) \) consists of a unique point, which is not contained in \( C \), by Fact 1(3) and that the set \( C \cap \overline{P_2P_2} \) contains two or more distinct points. Since the set \( C \cap \sigma(L_{\tau_2}) \) consists of exactly \( d \) points for any \( \sigma \in K_{P_2} \), the number of ramification points in \( Z \neq 0 \) is exactly \( p^e \times d \). On the other hand, for each \( b \in K \) with \( h'(b) = 0 \), there exist exactly \( d \) points \((a, b)\) such that \( f(a, b) = 0 \), since \( z_0 = 0 \) and \( h(b) \neq 0 \) by Proposition 3. Therefore, \( h'(y) \) has exactly \( p^e \) roots. Let \( R \) be a ramification point of \( \pi_Q \) which is contained in \( Z \neq 0 \). Since \( R \) is tame (stated above), \( e_R \) is computed as the order of \( \frac{dx}{dy} = -\frac{h'(y)}{f_x} \) at \( R \) plus one. Since \( e_R = l \) for any ramification point \( R \in C \) with \( Z \neq 0 \), the polynomial \( h'(y) \) is divisible by \((y - b)^{l-1}\) if \( h'(b) = 0 \). Therefore, \( h'(y) \) should be of the form \( c(y - b_1)^{l-1} \cdots (y - b_{p^e})^{l-1} \), which is of degree \( p^e(l - 1) \). Since \( h'(y) \) is of degree \( p^e l - 2 \), by Proposition 3, we have \( p^e = 2 \). Since \( l \geq 3 \) divides \( p^e - 1 = 1 \), this is a contradiction.

**Proof of Theorem 2 (when \( l \geq 3 \)).** It follows from Lemma 10 that \( P_2 \neq Q \) and \( Q \notin C \). If \( l \geq 5 \) or \( l = 3 \), then this is a contradiction to Proposition 2(2)(4). Assume that \( l = 4 \). Then, by Proposition 2(3), there exists two distinct outer Galois points \( P_3, P_4 \) such that \( \gamma(P_4) = P_4 \) for any \( \gamma \in G_{P_3} \). Then, the point \( P_4 \) satisfies the condition of “\( Q \)” as in Proposition 2(1) for \( P_3 \). Then, this is a contradiction to Lemma 10. 

\( \square \)
6. Proof of Theorem 2 (The case where \( l \leq 2 \))

Let \( p \geq 3 \), let \( e \geq 1 \), let \( l \leq 2 \) and let \( C \) be a smooth plane curve of degree \( d = p^el \geq 4 \). We denote by \( L_\infty \subset \mathbb{P}^2 \) the line defined by \( Z = 0 \). Let \( P \in \mathbb{P}^2 \setminus C \) be Galois with respect to \( C \). Assume that \( P = (1 : 0 : 0) \). Let \( \gamma \in G_P \) and let \( A_\gamma \) be a \( 3 \times 3 \) matrix representing \( \gamma \). Then,

\[
A_\gamma = \begin{pmatrix}
a_{11}(\gamma) & a_{12}(\gamma) & a_{13}(\gamma) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( a_{11}(\gamma) = \pm 1 \) and \( a_{12}(\gamma), a_{13}(\gamma) \in K \). Then, \( \gamma^*(x) = a_{11}(\gamma)x + a_{12}(\gamma)y + a_{13}(\gamma) \). Note that \( K_P = \{ \gamma \in G_P | a_{11}(\gamma) = 1 \} \). Let \( g(x, y) := \prod_{\sigma \in K_P} (x + a_{12}(\sigma)y + a_{13}(\sigma)) \). Note that the set of roots \( \{ a_{12}(\sigma)y + a_{13}(\sigma) | \sigma \in K_P \} \subset K(y) \) forms an additive subgroup of \( K(y) \). According to [8, Proposition 1.1.5 and Theorem 1.2.1], \( g(x, y) \in K[y][x] \) has only terms of degree equal to some power of \( p \) in variable \( x \).

Therefore, \( g(x, y) = \alpha_0(y)x^p + \alpha_{p-1}(y)x^{p-1} + \cdots + \alpha_1(y)x + \alpha_0(y) \) for some \( \alpha_0(y), \ldots, \alpha_0(y) \in K[y] \) with deg \( \alpha_i(y) \leq p^e - p^i \) for \( i = 0, \ldots, e \). Then, \( \alpha_0(y) = 1 \) and \( \alpha_0(y) = \prod_{\sigma \in K_P \setminus 0} (a_{12}(\sigma)y + a_{13}(\sigma)) \).

Assume that \( l = 1 \). Then, \( K_P = G_P \) and \( \sigma \in K(y) \), since \( \sigma^* \sigma = \sigma \) for any \( \sigma \in G_P \). There exists \( h(y) \in K(y) \) such that \( g(x, y) + h(y) = 0 \) in \( K(x, y) \). Let \( h(y) = h_1(y)/h_2(y) \), where \( h_1, h_2 \in K[y] \). Then, \( g(x, y)h_2(y) + h_1(y) = 0 \) on \( C \). Let \( f(x, y) \) be a defining polynomial. Then, there exists \( v(x, y) \in K[x, y] \) such that \( f(x, y)v(x, y) = g(x, y)h_2(y) + h_1(y) \) as polynomials. Since \( (1 : 0 : 0) \notin C, f(x, y) \) has the term of degree \( p^e \) in variable \( x \). Comparing the coefficient of degree \( p^e \) in variable \( x \), we have \( v(x, y) \in K[y] \) and \( v(x, y) = h_2(y) \) up to a constant. Then, \( h_2(y) \) divides \( h_1(y) \) and we have \( h(y) \in K[y] \). Therefore, \( g(x, y) + h(y) \) is a defining polynomial.

**Lemma 11.** Assume that \( l = 1 \). Then, the defining equation of \( C \) is of the form \( g(x, y) + h(y) = 0 \), where \( g(x, y) \in K[y][x] \) has only terms of degree equal to some power of \( p \) in variable \( x \).

Assume that \( \delta'(C) \geq 2 \). Let \( P_2 \in \mathbb{P}^2 \setminus (C \cup \{ P \}) \) be Galois with respect to \( C \). By taking a suitable system of coordinates, we may assume that \( P_2 = (0 : 1 : 0) \). Then, \( \overline{PP_2} = L_\infty \). Similar to the previous section, we consider the group homomorphism \( r_P : G_P \to \text{Aut}(\overline{PP_2}) \), which is induced by the restriction. The cardinality of the kernel \( \text{Ker } r_P \) is a power of \( p \). We denote
it by $p^v$. Obviously, $0 \leq v \leq e$. Then, Ker $r_P = \{ \sigma \in G_P | \sigma(P_2) = P_2 \}$, since $P_2 \in L_o$ if and only if $\sigma(P_2) = P_2$ for $\sigma \in K_P$. Since $a_{12}(\sigma) = 0$ if and only if $\sigma \in K_P$, $\omega_0(y)$ is of degree $p^v - p^\nu$ in variable $y$.

**Lemma 12.** If $l = 1$, then $v = e$.

**Proof.** We assume that $v < e$. Then, the defining polynomial $g(x, y) + h(y)$ has the term $\omega_0(y)x$, which is of degree $p^v - p^\nu > 0$ in variable $y$. Since $P_2 = (0 : 1 : 0)$ is Galois, the defining polynomial $g(x, y) + h(y)$ has only terms of degree equal to some power of $p$ in variable $y$, by Lemma 11. Therefore, $p^v - p^\nu = p^v(p^{e-v} - 1)$ is a power of $p$. Then, $p^{e-v} - 1 = p^b$ for some integer $b$. This implies $b = 0$ and $p = 2$. This is a contradiction. □

By Lemmas 11 and 12, we have a defining equation $g(x) + h(y) = 0$, where $g, h$ have only terms of degree equal to some power of $p$. It is not difficult to check that $C$ is singular. This is a contradiction.

Assume that $l = 2$. Let $\tau \in G_P \setminus K_P$. Then, $\tau(x, y) = (-x + a_{12}(\tau)y + a_{13}(\tau), y)$ for some $a_{12}(\tau), a_{13}(\tau) \in K$. Then, $G_P = \{ \sigma \tau^i | \sigma \in K_P, i = 0, 1 \}$. Note that $\sigma\tau(x, y) = (-x + (a_{12}(\sigma) + a_{12}(\tau))y + (a_{13}(\sigma) + a_{13}(\tau)), y)$. Therefore, $\hat{g}(x, y) := \prod_{\gamma \in G_P} \gamma^g(x) = g(x, y) \times g(-x + a_{12}(\tau)y + a_{13}(\tau), y) = -g^2(x, y) - g(x, y)g(-a_{12}(\tau)y - a_{13}(\tau), y)$, since $g(x, y)$ is linear in variable $x$. Since $\gamma \hat{g}(x, y) = \hat{g}(x, y)$ for any $\gamma \in G_P$, there exists $h(y) \in K(y)$ such that $f(x, y) := \hat{g}(x, y) + h(y) = 0$ in $K(x, y)$. Then, $h(y)$ is a polynomial and $f(x, y)$ is a defining polynomial (similarly to the case $l = 1$).

**Lemma 13.** Assume that $l = 2$. Then, the defining equation of $C$ is of the form $g^2(x, y) + g(x, y)g(ay + b, y) + h(y) = 0$, where $a, b \in K$ and $g \in K[y][x]$ has only terms of degree equal to some power of $p$ in variable $x$.

We consider $P_2 = (0 : 1 : 0)$. It follows from Lemma 13 that there exist polynomials $g_1(x, y) \in K[x][y]$ and $h_1(x) \in K[x]$ such that $g_1(x, y)$ has only terms of degree equal to some power of $p$ in variable $y$ and $f_1(x, y) := g_1^2(x, y) + g_1(x, y)g_1(x, cx + d) + h_1(x)$ is a defining polynomial of $C$ for some $c, d \in K$. Let $g_1(x, y) = \beta_{e}(x)y^{p^v} + \beta_{e-1}(x)y^{p^{v-1}} + \cdots + \beta_0(x)y$, where $\beta_{e}(x) = 1$ and $\beta_{e-1}(x), \ldots, \beta_0(x) \in K[x]$. Since $f(x, y)$ and $f_1(x, y)$ are defining polynomials of $C$, we have $cf(x, y) = f_1(x, y)$ for some $c \in K$. 

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**Lemma 14.** If \( l = 2 \), then \( v = e \).

**Proof.** Assume that \( v < e \). Firstly we prove that \( p = 3 \) and \( v = e - 1 \).

Now, \( z_0(y) \) is of degree \( p^e - p^v > 0 \). Considering the polynomials \( g^2(x, y), g(x, y)g(a y + b, y) \) and \( h(y)\), \( f(x, y) \) has the term \( \frac{2}{x^2}y^2 \), which is of degree \( 2(p^e - p^v) \) in variable \( y \). We consider this term for \( f_1(x, y) \). Since \( g_1(x, y)g_1(x, cx + d) = \sum \beta_i(x)g_1(x, cx + d)y^{iv} \) has only terms of degree equal to some power of \( p \) in variable \( y \) and \( 2(p^e - p^v) \) is not a power of \( p \) in \( p > 2 \), the term of the highest degree of \( \frac{2}{x^2}y^2 \) does not appear here. Therefore, this term should appear in \( g^2_1(x, y) \) (up to a constant). Since the polynomial \( g^2_1(x, y) = \sum \beta_i(x)\beta_j(x)y^{iv + p^j} \) has only terms of degree \( p^i + p^j = p^i(1 + p^{i-j}) \) with \( i \leq j \) and \( 0 \leq i, j \leq e \) in variable \( y \), we have \( 2p^e - p^v - 1 = p^i(1 + p^{i-j}) \) for some \( i, j \) with \( i < j \). Then, we should have \( i = v \) and \( 2p^e - p^v - 1 = 1 + p^{j-i} \). This implies that \( 2p^e - p^v - p^{j-i} = 3 \). If \( j = i \), then \( p = 2 \). This is a contradiction. If \( j \neq i \), then \( p^{j-i}(2p^e - p^{j-i} - 1) = 3 \). We should have \( p = 3 \), \( j - i = 1 \) and \( p^e - v - 1 = 1 \). Since \( i = v \) and \( i < j \) as above, \( i = v = e - 1 \) and \( j = e \).

Secondly we prove that \( p = 3 \), \( e = 1 \) and \( v = 0 \). Note that \( p^e - p^{e-1} = 2p^e - 1 \) in \( p = 3 \). Since the polynomial \( g^2(x, y) = \sum \beta_i(x)\beta_j(x)y^{iv + p^j} \) has the term \( 2z_0(y)\beta_i(x)y^{iv + p^j} \), which is of degree \( 2p^e - 1 \) in variable \( y \), and the polynomial \( g_1(x, y)g_1(x, cx + d) \) has only terms of degree equal to some power of \( p \) in variable \( y \), the term of the highest degree \( 2p^e - 1 + p^e + 1 \) of \( 2z_0(y)\beta_i(x)y^{iv + p^j} \) appears in \( g^2_1(x, y) \) (up to a constant). Since \( g^2_1(x, y) = \sum \beta_i(x)\beta_j(x)y^{iv + p^j} \), and \( p^i + p^j = 2p^e - 1 \) implies that \( i = j = e - 1 \), \( \beta^{e-1}_i(x)y^{2p^e - 1} \) has the term of the highest degree \( 2p^e - 1 + p^e + 1 \) of \( 2z_0(y)\beta_i(x)y^{iv + p^j} \). Let \( k \) be the degree of \( \beta^{e-1}_i(x) \). Since \( \beta^{e-1}_i(x) \) has the term of degree at least \( (p^e + 1)/2 \), we have \( p^e + 1)/2 \leq k \leq p^e - p^{e-1} \). \( 2p^e - 1 \). Then, \( \beta^{e-1}_i(x)\beta_0(x)y^{iv + p^j} \) is of degree \( k + (p^e - p^{e-1}) = k + 2p^e - 1 \) in variable \( x \). Since \( (p^e + 1)/2 + (p^e - p^{e-1}) = p^e + (p^e - p^{e-1} + 1)/2 = p^e + (p^e - p^{e-1} + 1)/2 \), the term of the highest degree of \( \beta^{e-1}_i(x)\beta_0(x)y^{iv + p^j} \) appears in \( g^2_1(x, y) \). Since \( g^2(x, y) = \sum \beta_i(x)\beta_j(x)y^{iv + p^j} \), \( k + (p^e - p^{e-1}) = p^{i} + p^{j} \) for some \( i_1 \leq j_1 \). Since \( p^e + (p^e - 1)/2 \leq k + (p^e - p^{e-1}) = p^{i} + p^{j} \), we have \( j_1 = e \) and \( i_1 = e - 1 \). Therefore, \( k = 2p^e - 1 = p^e - p^{e-1} = p^e - p^v \). We have \( e = 1 \), since \( \deg \beta_i(x) = p^e - p^v \) if and only if \( i = 0 \).

Finally, we consider the remaining case where \( p = 3 \), \( e = 1 \) and \( v = 0 \). Then, \( g(x, y) = x^3 - (x y + \beta^2)x \) and \( g_1(x, y) = y^3 - (x + \beta)_y \) for some \( x, \beta, x_1, \beta_1 \in K \) with \( x, x_1 \neq 0 \). Note that \( g(a y + b) = (a y + b)((a + x)y + (b + \beta))((a - x)y + (b - \beta)) \). Since \( f(x, y) = g^2(x, y) = g^2_1(x, y) = g^2_2(x, y) \), we have \( f(x, y) = g^2(x, y) = g^2_1(x, y) = g^2_2(x, y) \).
\(g(ay + b, x)g(x, y) + h(y)\), the coefficient of \(xy^2\) of \(f(x, y)\) is equal to the one of \(g(ay + b, y)\), which is equal to \(x^2a(a + z)(a - z)\). If \(a(a + z)(a - z) \neq 0\), then the coefficient of \(xy^2\) of \(f_1(x, y)\) is not zero. However, by considering \(g_1^2(x, y) + g_1(x, cx + d)g_1(x, y)\), that is zero. Therefore, we should have \(a(a + z)(a - z) = 0\). Then, \(g(ay + b, y)\) is of degree at most two. If \(g(ay + b, y)\) is of degree at most one, then two of the three conditions \(a = 0, a + x = 0\) and \(a - x = 0\) hold. Then, we have \(a = x = 0\). This is a contradiction to \(x \neq 0\). Therefore \(g(ay + b, y)\) is of degree two. Then the coefficient of \(x^3y^2\) of \(f(x, y)\) is not zero. Since \(y^2\) appears only in \(g_1^2(x, y)\) or \(h_1(x, y)\) for \(f_1(x, y)\) and \(g_1^2(x, y) = y^6 - 2(x_1x + \beta_1)y^4 + (x_1x + \beta_1)^2y^2\), the coefficient of \(x^3y^2\) of \(f_1(x, y)\) is zero. This is a contradiction.

By Lemma 14, we have \(p^\nu = p^\epsilon\). Then, \(g(x, y) \in K[x]\) and \(g_1(x, y) \in K[y]\). We denote \(g(x, y)\) by \(g(x)\) and \(g_1(x, y)\) by \(g_1(y)\). We have \(f(x, y) = g_1^2(x) + g(x)g(ay + b) + \lambda_1g_1^2(y) + \lambda_2\) for some \(\lambda_1, \lambda_2 \in K\). Let \(G(X, Z) = Z^\nu g(X/Z)\) and let \(G_1(Y, Z) = Z^\nu g_1(Y/Z)\). Then, \(F(X, Y, Z) = Z^{2\nu} f(X/Z, Y/Z) = G^2(X, Z) + G(X, Z)(G(aY, Z) + g(b)Z^\nu) + \lambda_1 G_1^2(Y, Z) + \lambda_2 Z^{2\nu}\). Let \(x\) (resp. \(y\)) be the coefficient of \(XZ^{\nu-1}\) (resp. \(YZ^{\nu-1}\)) for \(G(X, Z)\) (resp. \(G_1(Y, Z)\)). Then, \(F_X = 2G(X, Z)zZ^{\nu-1} + xZ^{\nu-1}(G(aY, Z) + g(b)Z^\nu),\) \(F_Y = azZ^{\nu-1}G(X, Z) + 2\lambda_1 G_1(Y, Z)\beta Z^{\nu-1}\) and \(F_Z = -2G(X, Z)\). \(\alpha Z^{\nu-2} - xZ^{\nu-2}(G(aY, Z) + g(b)Z^\nu) + G(X, Z)(-a\beta YZ^{\nu-2}) - 2\lambda_1 G_1(Y, Z)\beta YZ^{\nu-2}\). Therefore, \(F_X(X, Y, 0) = F_Y(X, Y, 0) = F_Z(X, Y, 0) = 0\) and we have singular points on the line \(L_\infty\).

We have the assertion of Theorem 2.

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