On Quasi-Polarized Manifolds Whose Sectional Genus is Equal to the Irregularity

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ABSTRACT - Let $(X, L)$ be a quasi-polarized manifold of dimension $n$. In our previous paper, we proved that if $\dim X = 3$ and $h^0(L) \geq 2$, then $g(X, L) \geq h^1(\mathcal{O}_X)$ holds. Here $g(X, L)$ denotes the sectional genus of $(X, L)$. In this paper, we give the classification of quasi-polarized 3-folds $(X, L)$ with $h^0(L) \geq 3$ and $g(X, L) = h^1(\mathcal{O}_X)$. Moreover as an application of this result, we also give the classification of polarized manifolds $(X, L)$ with $\dim Bs|L| = 1$, $h^0(L) \geq n$ and $g(X, L) = h^1(\mathcal{O}_X)$.

1. Introduction.

Let $(X, L)$ be a quasi-polarized manifold with $\dim X = n$. For this pair $(X, L)$, the sectional genus $g(X, L)$ is defined by the following formula:

$$g(X, L) = 1 + \frac{1}{2}(K_X + (n - 1)L)L^{n-1},$$

where $K_X$ is the canonical bundle of $X$. Then there is the following conjecture which was proposed by Fujita [7, (13.7) Remark].

**Conjecture 1.1 (Fujita).** Let $(X, L)$ be a quasi-polarized manifold. Then $g(X, L) \geq q(X)$, where $q(X) := \dim H^1(\mathcal{O}_X)$ is the irregularity of $X$.

For this conjecture, there are some results (see [9], [10], [12] and so on). But it is unknown whether this conjecture is true or not even for the case of $\dim X = 2$. If $\dim X = 2$, then this conjecture is true if $h^0(L) > 0$ (see [9]). Moreover the classification of quasi-polarized surfaces $(X, L)$ with $g(X, L) = q(X)$ and $h^0(L) \geq 1$ was obtained (see [8], [9]).

If $\dim X = 3$ and $h^0(L) \geq 2$, it is known that this conjecture is true, and

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the classification of polarized 3-folds \((X, L)\) with \(g(X, L) = q(X)\) and \(h^0(L) \geq 3\) was given (see [12]).

In this paper, we will give the classification of quasi-polarized 3-folds with \(g(X, L) = q(X)\) and \(h^0(L) \geq 3\). As an application of this result, we are able to give the classification of polarized \(n\)-fold \((X, L)\) with \(g(X, L) = q(X)\), \(\dim \text{Bs}[L] = 1\) and \(h^0(L) \geq n\). (Here we note that \(g(X, L) \geq q(X)\) holds if \(\dim \text{Bs}[L] = 1\).)

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2. Preliminaries.

**Definition 2.1.** Let \(X\) and \(Y\) be projective varieties with \(\dim X > \dim Y \geq 1\), and let \(f : X \to Y\) be a surjective morphism with connected fibers. Then \((f, X, Y)\) is called a fiber space. Moreover if \(L\) is a nef and big (resp. an ample) line bundle on \(X\), then \((f, X, Y, L)\) is called a quasi-polarized (resp. polarized) fiber space.

**Lemma 2.1.** Let \(X\) and \(C\) be smooth projective varieties with \(\dim X = n\) and \(\dim C = 1\), and let \(L\) be a nef and big line bundle on \(X\). Assume that there exists a fiber space \(f : X \to C\) such that \(h^0(K_F + L_F) \neq 0\) for a general fiber \(F\) of \(f\). Then \(f^*(K_{X/C} + L)\) is ample.

**Proof.** First we note that there exists a natural number \(m\) such that \((mL)^n - n(mL)^{n-1}F > 0\). Then by [3, Lemma 4.1], there exists a natural number \(k\) such that \(\mathcal{O}_X(k(mL - F))\) has a nontrivial global section. Hence we have an injective map \(\mathcal{O}_X(kF) \to \mathcal{O}(kmL)\). On the other hand, there exists a line bundle \(\mathcal{N}\) on \(C\) such that \(\mathcal{O}(kF) = f^*(\mathcal{N})\). Hence by [4, Corollary 1.9] we see that \(f^*(K_{X/C} + L)\) is ample and we get the assertion. \(\square\)

**Definition 2.2.** (i) Let \((X_1, L_1)\) and \((X_2, L_2)\) be quasi-polarized varieties. Then \((X_1, L_1)\) and \((X_2, L_2)\) are said to be birationally equivalent if there is another variety \(G\) with birational morphisms \(g_i : G \to X_i\) \((i = 1, 2)\) such that \(g_1^*L_1 = g_2^*L_2\).

(ii) Let \((f_1, X_1, Y, L_1)\) and \((f_2, X_2, Y, L_2)\) be quasi-polarized fiber spaces. Then \((f_1, X_1, Y, L_1)\) and \((f_2, X_2, Y, L_2)\) are said to be birationally equivalent if there is another variety \(G\) with birational morphisms \(g_i : G \to X_i\) \((i = 1, 2)\) such that \(g_1^*L_1 = g_2^*L_2\) and \(f_1 \circ g_1 = f_2 \circ g_2\).
DEFINITION 2.3. Let $X$ be a normal projective variety of dimension $n$ and let $D$ be a $\mathbb{Q}$-divisor on $X$. Then $D$ is said to be \textit{generically nef} if $DL_1 \cdots L_{n-1} \geq 0$ for any collection of ample Cartier divisors $L_1, \ldots, L_{n-1}$ on $X$.

DEFINITION 2.4. Let $(X, L)$ be a quasi-polarized variety of dimension $n$. Then the $\Delta$-\textit{genus} $\Delta(X, L)$ of $(X, L)$ is defined by the following:

$$\Delta(X, L) = n + L^n - h^0(L).$$

PROPOSITION 2.1. Let $(X, L)$ be a quasi-polarized manifold of dimension $n$. If $K_X + (n-1)L$ is not generically nef, then $\Delta(X, L) = 0$ or $(X, L)$ is birationally equivalent to a scroll over a smooth curve.

PROOF. See [16, Proposition 1.3]. \qed

3. Main results.

First we will prove the following theorem.

THEOREM 3.1. Let $(f, X, C, L)$ be a quasi-polarized fiber space such that $X$ and $C$ are smooth with $\dim X = n$ and $\dim C = 1$. Then $g(X, L) \geq g(C)$. Moreover if $g(X, L) = g(C)$, then $(X, L)$ is one of the following two types.

(a) $\Delta(X, L) = 0$.

(b) The pair $(X, L)$ is birationally equivalent to a scroll over $C$.

PROOF. (1) If $g(C) = 0$, then $g(X, L) \geq 0 = g(C)$ by [16, Theorem 1.1]. Moreover if $g(X, L) = 0 = g(C)$, then by [16, Theorem 1.2] we have $\Delta(X, L) = 0$.

(2) Next we assume that $g(C) \geq 1$.

(2.1) First we assume that $K_X + (n-1)L$ is generically nef. Then by [15, 1.2 Theorem] we see that there exists a natural number $j$ with $1 \leq j \leq n-1$ such that $h^0(K_X + jL) > 0$. Hence $h^0(K_F + jL_F) > 0$ for any general fiber $F$ of $f$. Then $f_*(K_{X/C} + jL) \neq 0$. By Lemma 2.1 we see that $f_*(K_{X/C} + jL)$ is ample. By the same argument as [10, Lemma 1.4.1], we get $(K_{X/C} + jL)L^{n-1} > 0$. Since $1 \leq j \leq n-1$, we have $(K_{X/C} + (n-1)L)L^{n-1} > 0$. Then

$$g(X, L) = g(C) + \frac{1}{2}(K_{X/C} + (n-1)L)L^{n-1} + (g(C) - 1)(L_{F})^{n-1} - 1 > g(C).$$
(2.2) Next we assume that $K_X + (n-1)L$ is not generically nef. Then by Proposition 2.1 we see that $\mathcal{A}(X, L) = 0$ or there exist a quasi-polarized variety $(X', L')$, a smooth projective variety $M$ and birational morphisms $\mu_1 : M \to X$ and $\mu_2 : M \to X'$ such that $(X', L')$ is a scroll over a smooth curve.

If $\mathcal{A}(X, L) = 0$, then we infer that $h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X) = 0$ (see [12, Lemma 1.15]). Hence $g(C) = 0$ and this contradicts the assumption of $g(C) > 0$. So we may assume that $(X', L')$ is a scroll over a smooth curve $B$. Let $f' : X' \to B$ be its fibration and let $h := f' \circ \mu_2 : M \to B$. Then for any general fiber $F_h$ of $h$, we have $h^1(\mathcal{O}_{F_h}) = 0$. Since $g(C) > 0$, we see that $f \circ \mu_1(F_h)$ is a point. Therefore by [2, Lemma 4.1.13] there exists a surjective morphism $\delta : B \to C$ such that $f \circ \mu_1 = \delta \circ h$. But since $f$ and $f'$ have connected fibers, we see that $\delta$ is an isomorphism. On the other hand, we can easily check that $g(X', L') = g(B)$. So we get $g(X, L) = g(X', L') = g(B) = g(C)$. Therefore we get the assertion. \hfill \Box

Remark 3.1. There exists an example of a quasi-polarized fiber space $(f, X, C, L)$ such that $g(X, L) = g(C)$ and $(X, L)$ is birationally equivalent to $(V, H)$ with $\mathcal{A}(V, H) = 0$. For example, let $(V, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Then we can easily see that $\mathcal{A}(V, H) = 0$. We take two general members $H_1$ and $H_2$ in $|H|$ and let $\mathcal{A}$ be a pencil which is generated by $H_1$ and $H_2$. By using this pencil, we can make a fiber space over a smooth curve. Namely, there exist a smooth projective variety $X$, a birational morphism $\mu : X \to \mathbb{P}^n$ and a fiber space $f : X \to C$ over a smooth curve $C$. We set $L := \mu^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Since $q(X) = 0$, we see that $C \cong \mathbb{P}^1$. Moreover $g(X, L) = g(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = 0 = g(C)$ and $(X, L)$ is birationally equivalent to $(V, H)$.

Next we consider quasi-polarized manifolds $(X, L)$ with $\dim X = 3$, $h^0(L) \geq 3$ and $g(X, L) = q(X)$.

Theorem 3.2. Let $(X, L)$ be a quasi-polarized 3-fold. Assume that $h^0(L) \geq 3$. If $g(X, L) = q(X)$, then $(X, L)$ satisfies one of the following two types.

(a) $\mathcal{A}(X, L) = 0$.

(b) The pair $(X, L)$ is birationally equivalent to a scroll over a smooth curve $C$.

Proof. By [6, Theorem 4.2], there exists a quasi-polarized variety $(X', L')$ which is birationally equivalent to $(X, L)$ and satisfies one of the
following conditions:

(i) $K_X + 2L'$ is nef for the canonical $\mathbb{Q}$-bundle $K_X$;
(ii) $\Delta(X, L) = \Delta(X', L') = 0$;
(iii) $(X', L')$ is a scroll over a curve,

where $X'$ is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities. Since $g(X, L) = g(X', L')$ and $q(X) = q(X')$, we may assume that $X$ has only $\mathbb{Q}$-factorial terminal singularities and $(X, L)$ satisfies one of the above conditions.

If $(X, L)$ is the type (ii), then $g(X, L) = 0$ by [6, (1.7) Corollary] and $q(X) = 0$ by [12, Lemma 1.15]. Hence we obtain $g(X, L) = q(X)$ in this case.

If $(X, L)$ is the type (iii), then we can check $g(X, L) = q(X)$ by easy calculation.

So we may assume that $K_X + 2L$ is nef. Let $\pi : \tilde{X} \to X$ be a resolution of $X$ such that $\tilde{X} / \pi^{-1}(\text{Sing}(X)) \cong X / \text{Sing}(X)$, and $\tilde{L} = \pi^*(L)$. Then $h^0(\tilde{L}) = h^0(L) \geq 3$. Let $A$ be a linear pencil which is contained in $[\tilde{L}]$ such that $A = A_M + Z$, where $A_M$ is the movable part of $A$ and $Z$ is the fixed part of $[\tilde{L}]$. We will make a fiber space by using this $A$. Let $\varphi : \tilde{X} \to \mathbb{P}^1$ be the rational map associated with $A_M$, and $\theta : \tilde{X}' \to \tilde{X}$ an elimination of indeterminacy of $\varphi$. So we obtain a surjective morphism $\varphi' : \tilde{X}' \to \mathbb{P}^1$. If necessary, we take the Stein factorization $\delta : C \to \mathbb{P}^1$ of $\varphi'$. Then we have a fiber space $f' : \tilde{X}' \to C$ such that $\varphi' \circ \delta = \varphi$. Let $f''$ be a general fiber of $f'$ and let $a := \text{deg} \delta$. We consider this quasi-polarized fiber space $(f', \tilde{X}', C, \theta^*(\tilde{L}))$. By the proof of [12, Theorem 2.1], we see that there exists a quasi-polarized fiber space $(f_1, X_1, C, L_1)$ which is birationally equivalent to $(f', \tilde{X}', C, \theta^*(\tilde{L}))$ such that $(f_1, X_1, C, L_1)$ satisfies one of the following conditions.

- $K_{X_1} + 2L_1$ is $f_1$-nef.
- $(f_1, X_1, C, L_1)$ is a scroll.

If $(f_1, X_1, C, L_1)$ is a scroll, then we see that $g(X, L) = q(X)$ and this is the type (b) in Theorem 3.2. So we may assume that $K_{X_1} + 2L_1$ is $f_1$-nef. In this case, by [14, Lemma 0.2], we see that $K_{X_1/C} + 2L_1$ is nef.

(a) The case of $g(C) \geq 1$. Then $\theta$ is the identity map. So we have $\tilde{X}' = \tilde{X}$ and $\theta^*(\tilde{L}) = \tilde{L}$. By the construction of the fiber space $(f', \tilde{X}', C, \theta^*(\tilde{L}))$, we get $\tilde{L} = \sum_{i=1}^{a} F_i + Z$, where each $F_i$ is a fiber of $f'$ and $Z$ is the fixed part of $[\tilde{L}]$. Then there exists an ample line bundle $P \in \text{Pic}(C)$ such that $\sum_{i=1}^{a} F_i = (f')^*(P)$. In particular $\text{deg} P = a$. 

CLAIM 3.1. \( a \geq 3 \).

PROOF. First we note that \( h^0(L) = h^0(\tilde{L}) = h^0(\sum_{i=1}^{a} F_i + Z) = h^0(\sum_{i=1}^{a} F_i) = h^0(P) \). Since \( h^0(L) \geq 3 \), we have \( h^0(P) \geq 3 \). If \( a \leq 2 \), then \( \Delta(C, P) = 1 + \deg P - h^0(P) = 1 + a - h^0(P) \leq 0 \). On the other hand, since \( P \) is an ample line bundle on \( C \), we have \( \Delta(C, P) \geq 0 \) by [5, Corollary 1.10] or [7, (4.2) Theorem]. Therefore \( \Delta(C, P) = 0 \). But then \( C \cong \mathbb{P}^1 \) (see [5, Lemma 3.1]) and this contradicts the assumption that \( g(C) \geq 1 \). Hence we have \( a \geq 3 \).

Here we note that \( \tilde{L} \) is numerically equivalent to \( aF' + Z \) by the construction above. By the same argument as in the proof of [12, Claim 2.2], we have

\[
(K_{X/C}^- + 2\tilde{L})(\tilde{L})^2 \geq t(K_{X/C}^- + 2\tilde{L})(\tilde{L})F'
\]

for any natural number \( t \) with \( t \leq a \). Hence \( (K_{X/C}^- + 2\tilde{L})(\tilde{L})^2 \geq 3(K_{X/C}^- + 2\tilde{L})(\tilde{L})F' \) holds because \( a \geq 3 \). Since \( g(C) \geq 1 \) and \( (\tilde{L}F')^2 \geq 1 \), we get

\[
g(\tilde{X}, \tilde{L}) = 1 + \frac{1}{2} (K_{X/C}^- + 2\tilde{L})(\tilde{L})^2
\]

\[
= g(C) + \frac{1}{2} (K_{X/C}^- + 2\tilde{L})(\tilde{L})^2 + (g(C) - 1)((L_{F'})^2 - 1)
\]

\[
\geq g(C) + \frac{3}{2} (K_{X/C}^- + 2\tilde{L})(\tilde{L})F'
\]

\[
= g(C) + 3g(F', \tilde{L}_{|F'}) + \frac{3}{2} (\tilde{L})^2 F' - 3.
\]

Since \( h^0(\tilde{L}_{|F'}) > 0 \) and \( \dim F' = 2 \) we have \( g(F', \tilde{L}_{|F'}) \geq q(F') \) by [9, Lemma 1.2 (2)]. Because \( g(C) + q(F') \geq q(\tilde{X}) \), we have

\[
g(\tilde{X}, \tilde{L}) \geq q(\tilde{X}) + 2g(F', \tilde{L}_{|F'}) + \frac{3}{2} (\tilde{L})^2 F' - 3.
\]

Since \( g(\tilde{X}, \tilde{L}) = g(X, L) = q(\tilde{X}) = q(\tilde{X}) \) holds, we get \( 2g(F', \tilde{L}_{|F'}) + \frac{3}{2} (\tilde{L})^2 F' - 3 \leq 0 \). Hence we have \( g(F', \tilde{L}_{|F'}) = 0 \). Therefore \( \kappa(F') = -\infty \) and by [9, Theorem 2.1] we have \( q(F') = 0 \). So \( q(\tilde{X}) = g(C) \) because \( g(C) = q(F') \geq g(\tilde{X}) \geq g(C) \). Hence we obtain \( g(\tilde{X}, \tilde{L}) = g(C) \), and by Theorem 3.1 we get the assertion in this case.

(b) The case of \( g(C) = 0 \). Let \( \gamma := \pi \circ \theta \).

(b.1) If \( a \geq 2 \), then
\[ g(X, L) = g(\widetilde{X}, \widetilde{L}) \\
   = g(\widetilde{X}', \theta^* (\widetilde{L})) \\
   = 1 + \frac{1}{2} \gamma^* (K_X + 2L)(\theta^* (\widetilde{L}))^2 \\
   \geq 1 + \gamma^* (K_X + 2L)(\theta^* (\widetilde{L}))F' \\
\]

because \( K_X + 2L \) is nef and \( a \geq 2 \). Let \( \tilde{D} := \theta(F') \). By [12, Claims 2.3 and 2.4], we have

\[ g(X, L) \geq 1 + \gamma^* (K_X + 2L)(\theta^* (\widetilde{L}))F' \\
   = 1 + \theta^*(\pi^*(\theta)(\widetilde{L}))F' \\
   \geq 1 + \theta^*(\tilde{K}_X + 2\tilde{L})(\theta^* (\widetilde{L}))F' \\
   \geq 1 + (\tilde{K}_X + F' + \theta^*(\widetilde{L}))(\theta^* (\widetilde{L}))F' \\
   = 2g(F', \theta^*(\widetilde{L})|_{F'}) - 1. \]

Since \( \dim F' = 2 \) and \( h^0(\theta^*(\widetilde{L})|_{F'}) > 0 \), we have \( g(F', \theta^*(\widetilde{L})|_{F'}) \geq q(F') \) by [9, Lemma 1.2 (2)]. Moreover since \( q(F') = q(F') + g(C) \geq q(X') = q(X) \), we get \( g(X, L) \geq 2q(X) - 1 \). Therefore \( q(X) \leq 1 \) because \( g(X, L) = q(X) \). In particular \( g(X, L) \leq 1 \). From [6, Corollaries (4.8) and (4.9)], we see that \( (X, L) \) is birationally equivalent to one of the types (a) and (b) in Theorem 3.2. (Here we use the assumption that \( g(X, L) = q(X) \).)

(b.2) Here we assume that \( a = 1 \). Then \( h^0(\theta^*(\widetilde{L})|_{F'}) \geq 2 \). By the same argument as in Case (2) in the proof of [12, Theorem 2.1] we have

\[ q(X) = g(X, L) \geq g(F', \theta^*(\widetilde{L})|_{F'}) \geq q(F') \geq q(X). \]

Hence we have \( \kappa(F') = -\infty \) by [9, Theorem 3.1] since \( g(F', \theta^*(\widetilde{L})|_{F'}) = q(F') \) and \( h^0(\theta^*(\widetilde{L})|_{F'}) \geq 2 \). Moreover we get

\[ q(\widetilde{X}') = q(X) = q(F'). \]

Here we apply the relatively minimal model theory for the fibration \( f' : \widetilde{X}' \to C \cong \mathbb{P}^1 \). Since \( \kappa(F') = -\infty \), we see that there exist smooth projective varieties \( X'' \) and \( T \) with \( \dim X'' = 3 \) and \( 1 = \dim C \leq \dim T \leq 2 \), a birational morphism \( \delta^* : X'' \to \widetilde{X}' \) and surjective morphisms \( \delta_1 : X'' \to T \) and \( \delta_2 : T \to C \) with connected fibers such that \( f' \circ \delta^* = \delta_2 \circ \delta_1 \) and \( F_{\delta_1} \) is birationally equivalent to a Fano manifold, where \( F_{\delta_1} \) is a general fiber of \( \delta_1 \). In particular \( q(F_{\delta_1}) = 0 \). We put \( f^* := f' \circ \delta^* \).
(b.2.1) Assume that \( \dim T = 1 \). Then \( \delta_2 \) is an isomorphism. Hence \( q(T) = 0 \) because \( C \cong \mathbb{P}^1 \). On the other hand, since \( q(F_{\delta_2}) = 0 \), we have \( q(X) = q(X^z) = q(T) = 0 \). Thus we get \( g(X, L) = 0 \) from the assumption that \( g(X, L) = q(X) \). Therefore by [6, (4.8) Corollary] we get the assertion.

(b.2.2) Next we assume that \( \dim T = 2 \). If \( q(X) \leq 1 \), then we have \( g(X, L) \leq 1 \) and by [6, Corollaries (4.8) and (4.9)] we get the assertion. So we may assume that \( q(X) \geq 2 \). Let \( F_{\delta_2} \) (resp. \( F^z \)) be a general fiber of \( \delta_2 \) (resp. \( f^z \)). Then
\[
\delta_1|_{F^z} : F^z \to F_{\delta_2}
\]
is a surjective morphism with connected fibers. Since a general fiber of \( \delta_1|_{F^z} \) is \( \mathbb{P}^1 \), we have \( q(F^z) = q(F_{\delta_2}) \). On the other hand, we have \( q(X^z) = q(T) \) because any general fiber of \( \delta_1 \) is \( \mathbb{P}^1 \), and we have \( q(F_{\delta_2}) = q(X^z) \) by (1). So we get \( q(T) = q(F^z) = q(F_{\delta_2}) = q(F_{\delta_2}) + q(\mathbb{P}^1) \). Now we are assuming that \( q(X^z) = q(X) \geq 2 \), so we have \( q(F_{\delta_2}) \geq 2 \). Therefore, considering the fiber space \( \delta_2 : T \to C \cong \mathbb{P}^1 \), we see from [1, Lemme] or [9, Lemma 1.5] that \( T \) is birationally equivalent to \( F_{\delta_2} \times \mathbb{P}^1 \). In particular \( \kappa(T) = -\infty \). So, taking the Albanese map of \( T \), there exists a morphism \( \alpha : T \to B \), where \( B \) is a smooth projective curve with \( g(B) = q(T) = q(F_{\delta_2}) \). Then \( \alpha \circ \delta_1 : X^z \to B \) has connected fibers. Moreover since \( q(X^z) = q(F^z) = q(F_{\delta_2}) = g(B) \) we obtain \( g(X^z, (\alpha \circ \delta_1)^* (\tilde{L})) = g(X, L) = q(X) = q(X^z) = g(B) \). Since \( (X^z, (\alpha \circ \delta_1)^* (\tilde{L})) \) is a quasi-polarized 3-fold, we get the assertion by Theorem 3.1. 

Here we want to propose the following conjecture which is a quasi-polarized manifolds' version of [12, Conjecture 2.15].

**Conjecture 3.1.** Let \( (X, L) \) be a quasi-polarized \( n \)-fold. Assume that \( h^0(L) \geq n \). If \( g(X, L) = q(X) \), then \( (X, L) \) is one of the following.

(a) \( \mathcal{A}(X, L) = 0 \).

(b) The pair \( (X, L) \) is birationally equivalent to a scroll over a smooth curve.

**Remark 3.2.** If \( n = 2 \) (resp. \( n = 3 \)), then this conjecture is true by [9, Theorem 3.1] (resp. Theorem 3.2 above).

Let \( (X, L) \) be a polarized manifold of dimension \( n \). If \( \text{Bs}|L| = \emptyset \) (resp. \( \dim \text{Bs}|L| = 0 \)), then by [2, Theorem 7.2.10] (resp. [11, Theorem 3.2]) we see that \( g(X, L) \geq q(X) \). Moreover, we can get a classification of \( (X, L) \) with \( g(X, L) = q(X) \) and \( \dim \text{Bs}|L| \leq 0 \) (see [17, (3.6) Theorem] and [11, Theorem 3.2]). So, as the next step, we consider the case where \( \dim \text{Bs}|L| = 1 \).
THEOREM 3.3. Let $(X, L)$ be a polarized manifold of dimension $n \geq 3$. Assume that $\dim \text{Bs}|L| = 1$.

(i) The inequality $g(X, L) \geq q(X)$ holds.
(ii) Furthermore we assume that $h^0(L) \geq n$. If $g(X, L) = q(X)$, then $(X, L)$ is one of the following.
(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(c) A scroll over a smooth curve.

PROOF. From the assumption, we see that there exist an $(n - 3)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-3}$ such that each $X_j$ is a normal and Gorenstein projective variety of dimension $n - j$ (see [13, Proposition 1.12 (2)]). Let $L_j = L_{X_j}$ for every $j$ with $1 \leq j \leq n - 3$. Then we see that

\[(2) \quad h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_1}) = \cdots = h^1(\mathcal{O}_{X_{n-3}})\]

and

\[(3) \quad g(X, L) = g(X_1, L_1) = \cdots = g(X_{n-3}, L_{n-3}).\]

Let $\pi : M_{n-3} \rightarrow X_{n-3}$ be a resolution of $X_{n-3}$. Then

\[(4) \quad g(M_{n-3}, \pi^*(L_{n-3})) = g(X_{n-3}, L_{n-3})\]

and

\[(5) \quad h^1(\mathcal{O}_{M_{n-3}}) \geq h^1(\mathcal{O}_{X_{n-3}}).\]

(i) Here we note that $h^0(L_{n-3}) \geq 2$. Hence by [12, Theorem 2.1] we have

\[(6) \quad g(M_{n-3}, \pi^*(L_{n-3})) \geq q(M_{n-3}).\]

Therefore by (2), (3), (4), (5) and (6), we get $g(X, L) \geq q(X)$.

(ii) Assume that $h^0(L) \geq n$. Then $h^0(L_{n-3}) \geq 3$. If $g(X, L) = q(X)$, then by (2), (3), (4), (5) and (6) we have $g(M_{n-3}, \pi^*(L_{n-3})) = q(M_{n-3})$ and $q(M_{n-3}) = q(X_{n-3})$. In particular, $X_{n-3}$ has the Albanese variety (see [2, Remark 2.4.2]). Let $\alpha : X_{n-3} \rightarrow \text{Alb}(X_{n-3})$ be its Albanese map, where $\text{Alb}(X_{n-3})$ is the Albanese variety of $X_{n-3}$. Then $\alpha \circ \pi : M_{n-3} \rightarrow \text{Alb}(X_{n-3})$ is the Albanese map of $M_{n-3}$. Since $(M_{n-3}, \pi^*(L_{n-3}))$ is a quasi-polarized 3-fold with $h^0(\pi^*(L_{n-3})) \geq 3$ and $g(M_{n-3}, \pi^*(L_{n-3})) = q(M_{n-3})$, we can apply Theorem 3.2. Then $(M_{n-3}, \pi^*(L_{n-3}))$ satisfies one of the following types:

- $\mathcal{A}(M_{n-3}, \pi^*(L_{n-3})) = 0$.
- $(M_{n-3}, \pi^*(L_{n-3}))$ is birationally equivalent to a scroll over a smooth curve.
If \(\mathcal{A}(M_{n-3}, \pi^*(L_{n-3})) = 0\), then \(g(X, L) = g(M_{n-3}, \pi^*(L_{n-3})) = 0\). Therefore we get the assertion from Fujita’s results (see [7, (12.1) Theorem and (5.10) Theorem]).

Next we assume that \((M_{n-3}, \pi^*(L_{n-3}))\) is birationally equivalent to a scroll over a smooth curve. Let \((V, H)\) be its scroll. If \(h^1(O_V) = 0\), then we see that \(g(X, L) = 0\) and we get the assertion. So we may assume that \(h^1(O_V) \geq 1\). Then the dimension of the image of Albanese map of \(V\) is one because \((V, H)\) is a scroll over a smooth curve. Since \(M_{n-3}\) and \(V\) are birationally equivalent each other, we see that the dimension of the image of \(\alpha \circ \pi\) is also one. Hence the dimension of the image of \(\alpha\) is also one. Since \(h^1(O_V) > 0\) implies \(h^1(O_X) > 0\), we can take the Albanese map \(\beta : X \rightarrow \text{Alb}(X)\) of \(X\).

**Claim 3.2.** \(\dim \beta(X) = 1\).

**Proof.** First we consider a map \(b : X_{n-3} \rightarrow X \rightarrow \text{Alb}(X)\). By the universality of the Albanese map, there exists a morphism \(c : \text{Alb}(X_{n-3}) \rightarrow \rightarrow \text{Alb}(X)\) such that \(c \circ \alpha = b\). On the other hand, since \(\dim \alpha(X_{n-3}) = 1\), we have \(\dim b(X_{n-3}) = \dim (c \circ \alpha)(X_{n-3}) \leq \dim \alpha(X_{n-3}) = 1\). But by [2, Propositions 5.1.1 and 5.1.2] we have \(\dim b(X_{n-3}) \geq 1\) because \(\dim \beta(X) \geq 1\). Hence \(\dim b(X_{n-3}) = 1\). Furthermore by using [2, Propositions 5.1.1 and 5.1.2], we also see \(\dim \beta(X) = 1\). \(\square\)

Since \(\dim \beta(X) = 1\), we find that \(\beta(X)\) is smooth and \(\beta : X \rightarrow \beta(X)\) is a fiber space over a smooth curve \(\beta(X)\). Let \(C = \beta(X)\). Since \(h^1(O_X) = g(C)\), we get \(g(X, L) = g(C)\). By [10, Theorem 1.4.2] we see that \((X, L)\) is a scroll over \(C\). So we get the assertion. \(\square\)

**Remark 3.3.** (i) Theorem 3.3 shows that [12, Conjecture 2.15] is true for the case of \(\dim \text{Bs}|L| = 1\).

(ii) If \(\dim \text{Bs}|L| \leq 0\), then we see that \(h^0(L) \geq n\). Hence by [17, (3.6) Theorem] and [11, Theorem 3.2] we infer that [12, Conjecture 2.15] is true for the case of \(\dim \text{Bs}|L| \leq 0\).

**References**


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