A Note on Posner’s Theorem with Generalized Derivations on Lie Ideals

V. De Filippis (*) - M. S. Tammam El-Sayiad (**)

Abstract - Let $R$ be a prime ring of characteristic different from 2, $Z(R)$ its center, $U$ its Utumi quotient ring, $C$ its extended centroid, $G$ a non-zero generalized derivation of $R$, $L$ a non-central Lie ideal of $R$. We prove that if $[[G(u), u], G(u)] \in Z(R)$ for all $u \in L$ then one of the following holds:
1. there exists $x \in C$ such that $G(x) = xx$, for all $x \in R$;
2. $R$ satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and there exist $a \in U$, $x \in C$ such that $G(x) = ax + xa + xx$, for all $x \in R$.

1. Introduction.

The motivation for this paper lies in an attempt to extend in some way the well known first Posner’s Theorem contained in [15]: there Posner proved that if $d$ is a derivation of a prime ring $R$ such that $[d(x), x]$ falls in the center of $R$, for all $x \in R$, then either $d = 0$ or $R$ is a commutative ring. Recently in [3] Cheng studied derivations of prime rings that satisfy certain special Engel type conditions: he showed that if $R$ is a prime ring of characteristic different from 2 and $d$ a non-zero derivation of $R$ which satisfies the condition $[[d(x), x], d(x)] = 0$ for all $x \in R$, then $R$ must be commutative.

Our purpose here is to continue this line of investigation by studying the set $S = \{[[G(x), x], G(x)], x \in L\}$, where $G$ is a generalized derivation

(*) Indirizzo dell’A.: D.I.S.I.A., Faculty of Engineering University of Messina, Contrada di Dio, 98166, Messina, Italy.
E-mail: defilippis@unime.it

(**) Indirizzo dell’A.: Department of Mathematics, Faculty of Science Beni Suef University, Beni Suef, Egypt.
E-mail: m_s_tammam@yahoo.com

2000 Mathematics Subject Classification. 16N60, 16W25.
defined on \( R \) and \( L \) is a non-central Lie ideal of \( R \). More specifically an additive map \( G : R \rightarrow R \) is said to be a generalized derivation if there is a derivation \( d \) of \( R \) such that, for all \( x, y \in R \), \( G(xy) = G(x)y + xd(y) \). A significative example is a map of the form \( G(x) = ax + xb \), for some \( a, b \in R \); such generalized derivations are called inner. Our goal is to confirm that there is a relationship between the structure of the prime ring \( R \) and the behaviour of suitable additive mappings defined on \( R \) that satisfy certain special identities. We will show that if any element of \( S \) is central in \( R \), then some informations about the form of the generalized derivation \( G \) and the structure of \( R \) can be obtained. More precisely we will prove the following:

**Theorem.** Let \( R \) be a prime ring of characteristic different from 2, \( Z(R) \) its center, \( U \) its Utumi quotient ring, \( C \) its extended centroid, \( G \) a non-zero generalized derivation of \( R \), \( L \) a non-central Lie ideal of \( R \). We prove that if \([[(G(u), u), G(u)] \in Z(R)\) for all \( u \in L \) then one of the following holds:

1. there exists \( \alpha \in C \) such that \( G(x) = \alpha x \), for all \( x \in R \);
2. \( R \) satisfies the standard identity \( s_4(x_1, \ldots, x_4) \) and there exist \( a \in U, \alpha \in C \) such that \( G(x) = ax + xa + \alpha x \), for all \( x \in R \).

For sake of clearness we premit the following:

**Fact 1.** Denote by \( T = U \ast_C C\{X\} \) the free product over \( C \) of the \( C \)-algebra \( U \) and the free \( C \)-algebra \( C\{X\} \), with \( X \) a countable set consisting of non-commuting indeterminates \( \{x_1, \ldots, x_n, \ldots\} \). The elements of \( T \) are called generalized polynomial with coefficients in \( U \). Moreover if \( I \) is a non-zero ideal of \( R \), then \( I, R \) and \( U \) satisfy the same generalized polynomial identities with coefficients in \( U \). For more details about these objects we refer the reader to [1] and [4].

**Fact 2.** Let \( a_1, \ldots, a_k \in U \) be linearly independent over \( C \) and \( a_1g_1(x_1, \ldots, x_n) + \ldots + a_kg_k(x_1, \ldots, x_n) = 0 \in T \), for some \( g_1, \ldots, g_k \in T = U \ast_C C\{X\} \). As a consequence of the result in [4], if for any \( i \), \( g_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_jh_j(x_1, \ldots, x_n) \) and \( h_j(x_1, \ldots, x_n) \in T \), then \( g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n) \) are the zero element of \( T \). The same conclusion holds if \( g_i(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T \), and \( g_i(x_1, \ldots, x_n) = \sum_{j=1}^n h_j(x_1, \ldots, x_n)x_j \) for some \( h_j(x_1, \ldots, x_n) \in T \).
2. The case of Inner Generalized Derivations.

In this section we study the case when the generalized derivation $G$ is inner defined as follows: $G(x) = ax + xb$ for all $x \in R$, where $a, b$ are fixed elements of $U$.

In all that follows we denote

$$P(x_1, x_2, x_3) = \left[ a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2] \right], a[x_1, x_2] + [x_1, x_2]b \right] x_3$$

$$- x_3 \left[ a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2] \right], a[x_1, x_2] + [x_1, x_2]b \right]$$

and assume that $R$ satisfies the generalized identity $P(x_1, x_2, x_3)$.

**Lemma 1.** If $R$ does not satisfy any non trivial generalized polynomial identity, then $a, b \in C$ and $G(x) = ax$, for all $x \in R$ and for $a = a + b$.

**Proof.** Denote by $T = U \ast_C C\{x_1, x_2, x_3\}$ the free product over $C$ of the $C$-algebra $U$ and the free $C$-algebra $C\{x_1, x_2, x_3\}$. Any element of $T$ is a generalized polynomial with coefficients in $U$.

Suppose that $R$ does not satisfy any non trivial generalized polynomial identity. Thus

$$P(x_1, x_2, x_3) = \left[ a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b \right] x_3$$

$$- x_3 \left[ a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b \right]$$

$$= a \left( [x_1, x_2]^2 a[x_1, x_2] + [x_1, x_2]^2 b - [x_1, x_2]a[x_1, x_2]^2 \right.$$

$$- [x_1, x_2]^2 (b - a)[x_1, x_2] + [x_1, x_2]^2 b \big) x_3$$

$$+ [x_1, x_2] (b - a)[x_1, x_2] a[x_1, x_2] + (b - a) [x_1, x_2]^2 b - [x_1, x_2] ba[x_1, x_2]$$

$$- [x_1, x_2] b(x_1, x_2) b - b[a(x_1, x_2)]^2 - b[x_1, x_2] (b - a)[x_1, x_2] + b(x_1, x_2)^2 b \big) x_3$$

$$- x_3 \left[ a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b, a[x_1, x_2] + [x_1, x_2]b \right] = 0 \in T.$$

Suppose that $\{1, a\}$ are linearly $C$-independent. Since $P(x_1, x_2, x_3)$ is a trivial generalized polynomial identity for $R$, then

$$\left( [x_1, x_2]^2 a[x_1, x_2] + [x_1, x_2]^2 b - [x_1, x_2]a[x_1, x_2]^2 - [x_1, x_2]^2 (b - a)[x_1, x_2]$$

$$+ [x_1, x_2]^2 b \big) x_3 = 0 \in T.$$
that is
\[ [x_1, x_2]^2 a[x_1, x_2] + [x_1, x_2]^3 b - [x_1, x_2] a[x_1, x_2]^2 \\
- [x_1, x_2]^2 (b - \alpha) [x_1, x_2] + [x_1, x_2]^3 b = 0 \in T. \]

This implies that \( \{1, b\} \) are linearly \( C \)-dependent. In fact, if not it follows that \( [x_1, x_2]^3 \) is an identity for \( R \), a contradiction. Thus \( b = \beta \in C \) and \( R \) satisfies
\[ [x_1, x_2]^2 a[x_1, x_2] + \beta [x_1, x_2]^3 - [x_1, x_2] a[x_1, x_2]^2 + [x_1, x_2]^2 a[x_1, x_2] \]
which is a non-trivial generalized identity, since we suppose that \( \{1, a\} \) are linearly \( C \)-dependent. This contradiction says that \( \{1, a\} \) are linearly \( C \)-dependent, that is \( a = \alpha \in C \).
Therefore \( R \) satisfies the generalized identity
\[ \left( [x_1, x_2](b + \alpha)[x_1, x_2] - [x_1, x_2]^2 (b + \alpha), [x_1, x_2](b + \alpha) \right) x_3 \\
- x_3 \left( [x_1, x_2](b + \alpha)[x_1, x_2] - [x_1, x_2]^2 (b + \alpha), [x_1, x_2](b + \alpha) \right) \]
that is
\[ 0 = \left( [x_1, x_2] b'[x_1, x_2]^2 b' - [x_1, x_2]^2 b'[x_1, x_2] b' - [x_1, x_2] b'[x_1, x_2] b'[x_1, x_2] \\
+ [x_1, x_2] b'[x_1, x_2]^2 b' \right) x_3 \\
- x_3 \left( [x_1, x_2] b'[x_1, x_2]^2 b' - [x_1, x_2]^2 b'[x_1, x_2] b' - [x_1, x_2] b'[x_1, x_2] b'[x_1, x_2] \\
+ [x_1, x_2] b'[x_1, x_2]^2 b' \right) \]
\[ = \left( [x_1, x_2] b'[x_1, x_2]^2 b' - [x_1, x_2]^2 b'[x_1, x_2] b' - [x_1, x_2] b'[x_1, x_2] b'[x_1, x_2] \\
+ [x_1, x_2] b'[x_1, x_2]^2 b' \right) x_3 \\
- x_3 [x_1, x_2] b'[x_1, x_2] b'[x_1, x_2] - x_3 \left( 2[x_1, x_2] b'[x_1, x_2]^2 - [x_1, x_2]^2 b'[x_1, x_2] \right) b' \]
where \( b' = b + \alpha \). If \( \{1, b'\} \) are linearly \( C \)-dependent, then
\[ -x_3 \left( 2[x_1, x_2] b'[x_1, x_2]^2 - [x_1, x_2]^2 b'[x_1, x_2] \right) \]
is a non-trivial generalized identity for \( R \), a contradiction. Then \( \{1, b'\} \) are linearly \( C \)-dependent, that is \( b' \in C \) as well as \( b \), and we are done. \( \square \)

**Lemma 2.** Let \( R = M_m(F) \) be the ring of \( m \times m \) matrices over the field \( F \) of characteristic different from 2, with \( m > 1 \), \( a, b \) elements of \( R \) such that
\[ [[au + ub, u], au + ub] \in Z(R) \]
for all \( u \in [R, R] \). Then one of the following holds:

1) \( a, b \in Z(R) \);
2) \( a - b \in Z(R) \) and \( m = 2 \).

**Proof.** The first aim is to prove that \( a - b \) is a diagonal matrix. Say \( a = \sum a_{ij}e_{ij}, b = \sum b_{ij}e_{ij} \), where \( a_{ij}, b_{ij} \in F \), and \( e_{ij} \) are the usual matrix units. Let \( u = [r_1, r_2] = [e_{ii}, e_{ij}] = e_{ij} \), for any \( i \neq j \). Thus

\[
[(ae_{ij} + e_{ij}b, e_{ij}), ae_{ij} + e_{ij}b] = (b_{ji} - a_{ji})(a_{ji} - b_{ji})e_{ij} \in Z(R)
\]

that is all the off-diagonal entries of the matrix \( a - b \) are zeros.

Let now \( \chi \in Aut_F(R) \) with \( \chi(x) = (1 + e_{ji})x(1 - e_{ji}) \). Of course

\[
[[\chi(a)u + u\chi(b), u], \chi(a)u + u\chi(b)] \in Z(R), \text{ for all } u \in [R, R].
\]

By calculation we have that

\[
\chi(a) = a + e_{ji}a - ae_{ji} - e_{ji}ae_{ji};
\]

\[
\chi(b) = b + e_{ji}b - be_{ji} - e_{ji}be_{ji};
\]

and by the previous argument we also have that \( \chi(a - b) \) is a diagonal matrix. In particular the \((j, i)\)-entry of \( \chi(a - b) \) is zero, that is \( a_{ii} - b_{ii} = a_{ij} - b_{ij} \). By the arbitrariness of \( i \neq j \), we have that \( a - b = z \) is a central matrix in \( R \) and \([au + ua + xu, u], au + ua + xu] \in Z(R), \text{ for all } u \in [R, R], \) that is \( R \) satisfies

\[
[[a[x_1, x_2]^2 - [x_1, x_2]^2a, a[x_1, x_2] + [x_1, x_2][a + z[x_1, x_2]]], x_3].
\]

In case \( m = 2 \) we are done. Thus assume that \( m \geq 3 \).

Suppose that \( a \) is not a diagonal matrix, for example let \( a_{ji} \neq 0 \) for \( i \neq j \).

Let now \( v = [e_{ii}, e_{ij} + e_{ji}] = e_{ij} - e_{ji} \). Thus

\[
X = [av^2 - v^2a, av + va + xv] \in Z(R)
\]

hence for any \( k \neq i, j \), the \((k, i)\)-entry \( X_{ki} \) of the matrix \( X \) is zero. By calculations we have that

(1) \( X_{ki} = a_{ki}(a_{ij} + a_{ji}) + a_{kj}(a_{ji} + a_{ii} + z) = 0 \)

On the other hand for \( w = [e_{ii}, e_{ij} - e_{ji}] = e_{ij} + e_{ji} \) we have

\[
Y = [aw^2 - w^2a, aw + wa + zw] \in Z(R) = 0.
\]

Again the \((k, i)\)-entry \( Y_{ki} \) of the matrix \( Y \) is zero, that is

(2) \( Y_{ki} = a_{ki}(a_{ij} + a_{ji}) + a_{kj}(a_{ji} + a_{ii} + z) = 0 \)
By (1) and (2) it follows that

\[ -2a_{ki}a_{ji} = 0 \]

Therefore we have that if \( a_{ji} \neq 0 \), then \( a_{ki} = 0 \) for all \( k \neq i, j \).

Let \( \varphi \in \text{Aut}_F(R) \) defined as \( \varphi(x) = (1 + e_{kj})x(1 - e_{kj}) \). Of course for all \( u \in [R, R] \), \( [\varphi(a)u^2 - u^2\varphi(a), \varphi(a)u + u\varphi(a) + xu] \in Z(R) \) Since the \((k, i)\)-entry of the matrix \( \varphi(a) \) is equal to \( a_{ji} \neq 0 \), then by the argument in (3) we have that the \((j, i)\)-entry \( a'_{ji} \) of the matrix \( \varphi(a) \) is zero. By calculations it follows \( 0 = a'_{ji} = a_{ji} \), a contradiction. Therefore \( a \) must be a diagonal matrix in \( R \). As above, for all \( r \neq s \), let \( \chi \in \text{Aut}_F(R) \) with \( \chi(x) = (1 + e_{sr})x(1 - e_{sr}) \). Hence also \( \chi(a) = a + e_{sr}a - ae_{sr} - e_{sr}ae_{sr} \) must be a diagonal matrix. In particular the \((s, r)\)-entry of \( \chi(a) \) is zero, that is \( a_{rr} = a_{ss} \).

By the arbitrariness of \( i \neq j \), we have that \( a \) is a central matrix in \( R \), and we are done again. \( \Box \)

**Proposition 1.** Let \( R \) be a prime ring of characteristic different from 2. Suppose that \( a, b \) are elements of \( U \) such that \( [(au + ub, u), au + bu] \in Z(R) \), for all \( u \in [R, R] \). Then one of the following holds:

1) \( a, b \in C \);
2) \( a - b \in C \) and \( R \) satisfies the standard identity \( s_4(x_1, \ldots, x_4) \).

**Proof.** By Lemma 1 we may assume that \( R \) satisfies the non-trivial generalized polynomial identity

\[
P(x_1, x_2, x_3) = \left[ a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2b, a[x_1, x_2] + [x_1, x_2]b \right] x_3
- x_3 \left[ a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2b, a[x_1, x_2] + [x_1, x_2]b \right].
\]

By a theorem due to Beidar (Theorem 2 in [1]) this generalized polynomial identity is also satisfied by \( U \). In case \( C \) is infinite, we have \( P(r_1, r_2, r_3) \in C \) for all \( r_1, r_2, r_3 \in U \otimes C \overline{C} \), where \( \overline{C} \) is the algebraic closure of \( C \). Since both \( U \) and \( U \otimes C \overline{C} \) are centrally closed ([6], Theorems 2.5 and 3.5), we may replace \( R \) by \( U \) or \( U \otimes C \overline{C} \) according as \( C \) is finite or infinite. Thus we may assume that \( R \) is centrally closed over \( C \) which is either finite or algebraically closed. By Martindale’s theorem [14], \( R \) is a primitive ring having a non-zero socle \( H \) with \( C \) as the associated division ring. In light of Jacobson’s theorem ([10], pag 75) \( R \) is isomorphic to a dense ring of linear transformations on some vector space \( V \) over \( C \).

Assume first that \( V \) is finite-dimensional over \( C \). Then the density of \( R \) on \( V \) implies that \( R \cong M_k(C) \), the ring of all \( k \times k \) matrices over \( C \). Since \( R \) is not commutative we assume \( k \geq 2 \). In this case the conclusion follows by Lemma 2.
Assume next that $V$ is infinite-dimensional over $C$. As in lemma 2 in [16], the set $[R, R]$ is dense on $R$ and so from $P(r_1, r_2, r_3) \in Z(R)$, for all $r_1, r_2, r_3 \in R$, we have $[ar + rb, r] \in Z(R)$, for all $r \in R$. Due to the infinity-dimensionality, $R$ cannot satisfies any polynomial identity. In particular the non-zero ideal $H$ cannot satisfies $s_4(x_1, \ldots, x_4)$. Suppose that either $a \notin C$ or $b \notin C$, then at least one of them doesn’t centralize the non-zero ideal $H$ of $R$, and we will prove that this leads to a contradiction.

Hence we are supposing that there exist $h_1, h_2 \in H$ such that either $[a, h_1] \neq 0$ or $[b, h_2] \neq 0$ and there exist $h_3, h_4, h_5, h_6 \in H$ such that $s_4(h_3, \ldots, h_6) \neq 0$.

Let $e^2 = e$ any non-trivial idempotent element of $H$. For $r = exe$, with any $x \in R$, we have that $[aexe + exebe, exe] \in Z(R)$. By commuting with $(1 - e)$ and then right multiplying by $(1 - e)$ it follows $2(1 - e)a(exe)b(1 - e) = 0$. Since $\text{char}(R) \neq 2$, we have that either $(1 - e)ae = 0$ or $eb(1 - e) = 0$. If $(1 - e)ae = 0$ then $ae = eae$ and $bae = beae$. On the other hand, in case $eb(1 - e) = 0$, we get $eb = ebe$, and so $eba = abea$. In any case we notice that the ring $eRe$ satisfies the generalized identity $\left[\left([\text{exe}X + X(ebe), X], \text{exe}X + X(ebe)\right), Y\right]$.

By Litoff’s theorem in [7] there exists $e^2 = e \in H$ such that $h_1, h_2, h_3, h_4, h_5, h_6 \in eRe$, moreover $eRe$ is a central simple algebra finite dimensional over its center. Since $s_4(h_3, \ldots, h_6) \neq 0$, then $eRe \cong M_t(C)$, for $t \geq 3$. By the finite dimensional case, we have that $eae, ebe \in Z(eRe)$, but this contradicts with the choices of $h_1, h_2$ in $eRe$. \hfill $\square$

3. The proof of the Theorem.

In this final section we will make use of the result of Kharchenko [11] about the differential identities on a prime ring $R$. We refer to Chapter 7 in [2] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

It is well known that any derivation of a prime ring $R$ can be uniquely extended to a derivation of its Utumi quotients ring $U$, and so any derivation of $R$ can be defined on the whole $U$ ([2], pg. 87).

Now, we denote by $\text{Der}(Q)$ the set of all derivations on $Q$. By a derivation word we mean an additive map $\Delta$ of the form $\Delta = d_1d_2 \ldots d_m$, with each $d_i \in \text{Der}(Q)$. Then a differential polynomial is a generalized polynomial, with coefficients in $Q$, of the form $\Phi(\Delta_j(x_i))$ involving non-commutative indeterminates $x_i$ on which the derivations words $\Delta_j$ act as
unary operations. The differential polynomial $\Phi(\mathcal{A}(x))$ is said a differential identity on a subset $T$ of $Q$ if it vanishes for any assignment of values from $T$ to its indeterminates $x_i$.

Let $D_{\text{int}}$ be the $C$-subspace of Der$(Q)$ consisting of all inner derivations on $Q$ and let $d$ and $\delta$ be two non-zero derivations on $R$. As a particular case of Theorem 2 in [11] we have the following result (see also Theorem 1 in [13]):

**FACT 3.** If $d$ is a non-zero derivation on $R$ and $\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$ is a differential identity on $R$, then one of the following holds:

1) either $d \in D_{\text{int}}$;

2) or $R$ satisfies the generalized polynomial identity

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Now we are ready to prove our main result:

**THEOREM.** Let $R$ be a prime ring of characteristic different from 2, $Z(R)$ its center, $U$ its Utumi quotient ring, $C$ its extended centroid, $G$ a non-zero generalized derivation of $R$, $L$ a non-central Lie ideal of $R$. We prove that if $[[G(u), u], G(u)] \in Z(R)$ for all $u \in L$ then one of the following holds:

1) there exists $x \in C$ such that $G(x) = xz$, for all $x \in R$;

2) $R$ satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and there exist $a \in U$, $x \in C$ such that $G(x) = ax + xa + za$, for all $x \in R$.

**PROOF.** By Theorem 3 in [12] every generalized derivation $g$ on a unital ring $R$ can be uniquely extended to the Utumi quotient ring $U$ of $R$, and thus we can think of any generalized derivation of $R$ to be defined on the whole $U$ and to be of the form $g(x) = ax + d(x)$ for some $a \in U$ and $d$ a derivation on $U$. Thus we will assume in all that follows that there exist $a \in U$ and $d$ derivation on $U$ such that $G(x) = ax + d(x)$. We note that we may assume that $R$ is not commutative, since $L$ is not central. Moreover, since $\text{char}(R) \neq 2$, there exists a non-central two-sided ideal $I$ of $R$ such that $[I, I] \subseteq L$ (see p. 4-5 in [8]; Lemma 2 and Proposition 1 in [5]). Therefore $[[G(u), u], G(u)] \in Z(R)$ for all $u \in [I, I]$. Moreover by [13] $R$ and $I$ satisfy the same differential polynomial identities, that is $[[G(u), u], G(u)] \in Z(R)$ for all $u \in [R, R]$. 
By assumption $R$ satisfies the differential identity

$$(4) \quad \left[ a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]], a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)] \right] x_3$$

$$- x_3 \left[ a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]], a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)] \right]$$

First suppose that $d$ is not an inner derivation on $U$. By Kharchenko’s theorem [11] $R$ satisfies the polynomial identity

$$(5) \quad \left[ a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]], a[x_1, x_2] + [y_1, x_2] + [x_1, y_2] \right] x_3$$

$$- x_3 \left[ a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]], a[x_1, x_2] + [y_1, x_2] + [x_1, y_2] \right]$$

in particular $R$ satisfies any blended component

$$\left[ a[x_1, x_2], [x_1, x_2]], a[x_1, x_2] \right] x_3 - x_3 \left[ a[x_1, x_2], [x_1, x_2]], a[x_1, x_2] \right]$$

that is

$$\left[ a[x_1, x_2], [x_1, x_2]], a[x_1, x_2] \right] \in Z(R)$$

and by Proposition 1 we have that $a = x \in C$. Thus from (5), it follows that $R$ satisfies the polynomial identity

$$\left[ ([y_1, x_2] + [x_1, y_2], [x_1, x_2]], x[x_1, x_2] + [y_1, x_2] + [x_1, y_2] \right] x_3$$

$$- x_3 \left[ ([y_1, x_2] + [x_1, y_2], [x_1, x_2]], x[x_1, x_2] + [y_1, x_2] + [x_1, y_2] \right].$$

Since $R$ satisfies a polynomial identity, there exists $M_k(F)$, the ring of all matrices over a suitable field $F$, such that $R$ and $M_k(F)$ satisfy the same polynomial identities (see [9], Theorem 2 p.54 and Lemma 1 p.89). For $x_1 = e_{22}, x_2 = e_{21}, y_1 = e_{21}$ and $y_2 = e_{12}$ we obtain

$$[y_1, x_2] = 0, \quad [x_1, y_2] = -e_{12}, \quad [x_1, x_2] = e_{21}$$

and it follows the contradiction

$$\left[ [-e_{12}, e_{21}], xe_{21} - e_{12} \right] = 2e_{12} + 2xe_{21} \notin Z(R).$$

Now consider the case when $d$ is an inner derivation induced by the element $b \in U$. Since $G(x) = ax + [b, x] = ax + bx - xb = (a + b)x + x(-b)$ and by Proposition 1, we have that either $a, b \in C$ or $a + 2b \in C$ and $R$ satisfies $s_4(x_1, \ldots, x_4)$. In the first case we conclude that $G(x) = ax$, for $a \in C$; in the second one $G(x) = a'x + xa' + x$, where $a' = -b$. \hfill \Box
REFERENCES


Manoscritto pervenuto in redazione il 9 giugno 2008.