Examples of Threefolds with Kodaira Dimension 1 or 2

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ABSTRACT - We construct three nonsingular threefolds \( X, X', \) and \( X'' \) with vanishing irregularities. \( X \) has Kodaira dimension \( \kappa(X) = 1 \) and its \( m \)-canonical transformation \( \varphi_{[mK_X]} \) has the following property: the minimum integer number \( m_0 \), such that the dimension of the image \( \dim \varphi_{[mK_X]}(X) = \kappa(X) = 1 \) for \( m \geq m_0 \), is given by \( m_0 = 32 \). \( X' \) and \( X'' \) have Kodaira dimension \( \kappa(X') = \kappa(X'') = 2 \) and their \( m \)-canonical transformations have the properties: \( \dim \varphi_{[mK_{X'}]}(X') = \kappa(X') = 2 \) if and only if \( m \geq 12 \), \( \dim \varphi_{[mK_{X''}]}(X'') = \kappa(X'') = 2 \) if and only if \( m = 9, 10 \) or \( m \geq 12 \).

Introduction.

One of the problems regarding the projective, algebraic, nonsingular variety \( X \), of dimension \( \dim X = d \) and of general type, is to establish the finiteness and also the birationality of the \( m \)-canonical transformation (improperly called a map) \( \varphi_{[mK_X]} : X \dashrightarrow \mathbb{P}^{P^1-1} \), where \( K_X \) is a canonical divisor on \( X \) and \( P_m \) is the \( m \)-genus of \( X \). In other words, the problem is to establish when the dimension of the image of \( X \) under \( \varphi_{[mK_X]} \) is \( d \) and, in addition, when \( X \) is birationally equivalent to its image \( \varphi_{[mK_X]}(X) \).

We want to generalize the above problem to any variety \( X \) with Kodaira dimension \( \kappa(X) > 0 \). Since “of general type” is equivalent to “\( \kappa(X) = d = \dim X \)”, the new problem is to establish when \( \dim \varphi_{[mK_X]}(X) = \kappa(X) \).

We indeed consider the following two problems:

(1) what is the minimum integer \( \mu_0 \) such that \( \dim \varphi_{[\mu_0K_X]}(X) = \kappa(X) \) for each \( m \geq \mu_0 \) and for each threefold \( X \) with Kodaira dimension \( \kappa(X) \)?

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(2) what is the maximum integer $M_0$ such that there exists a threefold $X$ with Kodaira dimension $\kappa(X)$ and $\dim \varphi_{|mK_X|}(X) < \kappa(X)$ for each $m < M_0$?

In particular, we are interested in these problems when $X$ moreover has vanishing irregularities.

First we recall the known answers to the analogous problems for surfaces.
When $X$ is a surface of general type, one has $\mu_0 = M_0 = 5$ (cf. F. Enriques [E], Cap. VIII, § 21; E. Bombieri [B]).

When $X$ is a surface with $\kappa(S) = 1$, it is known that $M_0 = 8$, as recently communicated by I. Dolgachev and proved by F. Catanese via email, and that $\mu_0 = 14$ (cf. F. Enriques [E], pp. 410–412, T. Katsura - K. Ueno [KU], F. Catanese - I. Bauer [CB]).

Consider now varieties of dimension $d \geq 3$.
C. Hacon - J. McMernan [HM], S. Takayama [Ta] and H. Tsuji [Ts] proved the existence of $\mu_0$ (depending only on $d$) for varieties of general type, but the value of $\mu_0$ is still unknown, even for $d = 3$. M. Chen [Che1, Che2, Che3] obtained upper bounds of $\mu_0$ for threefolds $X$ of general type, under some hypotheses on the geometric genus $p_g = P_1$, or on the $m$-genus $P_1, m > 1$, of $X$. In some cases, such limitations are optimal, thanks to examples due to Chen himself [Che2], S. Chiaruttini - R. Gattazzo [CG], S. Chiaruttini [Chi] and C. Hacon considering an example of Reid (cf. [Che3, Re]).

For threefolds $X$ with $\kappa(X) = 1$, it is known that there exists an effectively computable integer $m_1$ such that the $m_1$-canonical linear system $|m_1K_X|$ induces the Iitaka fibration for every threefold $X$ (cf. [FM], Corollary 6.2).

When $\kappa(X) = 2$, J. Kollár conjectures the existence of an integer $m_2$, independent of $X$, such that the $m$-canonical transformation $\varphi_{|mK_X|}$ gives the Iitaka fibration for $m \geq m_2$ [K, Remark 3.4]; concerning this conjecture cf. [P]. Moreover, Kollár conjectures that the $m$-genus of $X$ is $P_m(X) > 0$ for some $m < 24492$ and for every $X$ [K, Corollary-Conjecture 3.3].

In the present paper we give three examples of threefolds, $X$ in Chapter 1, $X'$ and $X''$ in Chapter 2, with vanishing irregularities. $X$ has Kodaira dimension 1 and its $m$-canonical transformation $\varphi_{|mK_X|}$ has the following property: the minimum integer $m_0$, such that $\dim \varphi_{|mK_X|}(X) = \kappa(X) = 1$ for $m \geq m_0$, is given by $m_0 = 32$. This implies that $\mu_0 \geq 32$ in problem (1) for threefolds with $\kappa = 1$ and vanishing irregularities.

The properties of $X$ (cf. Section 1.8) imply also that $M_0 \geq 20$ in problem (2) for threefolds with $\kappa = 1$ and vanishing irregularities.

$X'$ and $X''$ have Kodaira dimension 2 and their $m$-canonical transformations have the properties: $\dim \varphi_{|mK_{X'}|}(X') = \kappa(X') = 2$ if and only if
$m \geq 12; \dim \varphi_{|mK_X|}(X'') = \kappa(X'') = 2$ if and only if $m = 9, 10$ or $m \geq 12$. It follows that $\mu_0 \geq M_0 \geq 12$ in problems (1) and (2) for threefolds with $\kappa = 2$ and vanishing irregularities.

The irregularities and the first plurigenera of $X$, $X'$, $X''$ are as follows:

**Table 1. Irregularities and the first plurigenera of $X$, $X'$ and $X''$.**

|   | $q_1$ | $q_2$ | $p_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_{10}$ | $P_{11}$ | $P_{12}$ | $P_{13}$ | $P_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| $X'$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| $X''$ | 0 | 0 | 0 | 0 | 1 | 2 | 1 | 1 | 2 | 3 | 3 | 3 | 5 | 5 | 5 |

In our constructions, the ground field $k$ is an algebraically closed field of characteristic 0, which we may assume to be the field of complex numbers $\mathbb{C}$.

**1. Construction of $X$.**

1.1 – *Imposing singularities on a degree six hypersurface $V$ in $\mathbb{P}^4$.*

Let us indicate as $f_0(X_0, X_1, X_2, X_3, X_4)$ a form (a homogeneous polynomial) defining a hypersurface of degree six $V \subset \mathbb{P}^4$ with a triple point at each of the five vertices $A_0 = (1, 0, 0, 0, 0)$, $A_1 = (0, 1, 0, 0, 0)$, $A_2 = (0, 0, 1, 0, 0)$, $A_3 = (0, 0, 0, 1, 0)$, $A_4 = (0, 0, 0, 0, 1)$ of the fundamental pentahedron.

$$V : f_0(X_0, X_1, X_2, X_3, X_4) =$$

$$X_0^3(a_{33000}X_1^3 + \cdots) + X_1^3(a_{23100}X_0^2X_2 + \cdots) + X_2^3(\cdots) + X_3^3(\cdots) + X_4^3(\cdots) +$$

$$+ a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1X_2^2X_3 + \cdots + a_{00022}X_0^2X_2^2X_3^2X_4^2,$$

where $a_{ijklh} \in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^lX_4^m$.

We want to impose an infinitely near double surface $S_i$ at the point $A_i$, $i = 0, 1, 2, 3, 4$, in the first neighbourhood. The surface $S_i$ is locally isomorphic to a plane, according to our hypothesis on the singularities in $[S_1]$, Introduction and section 1.

We impose here a double surface $S_4$ infinitely near $A_4$ and after this, by means of a permutation of indices and variables, we impose the same singularity at the other $A_j$, $j < 4$. 
The permutations of the indices $ijklh$ of the coefficient $a_{ijklh}$ and of variables $X_0, \ldots, X_4$, which appear in $a_{ijklh}X_0^iX_1^jX_2^kX_3^lX_4^h$, passing from $A_4$ to $A_3$, from $A_3$ to $A_2$, from $A_2$ to $A_1$ and from $A_1$ to $A_0$, are as follows.

\[ \begin{align*}
& A_4 \mapsto A_3 \mapsto A_2 \mapsto A_1 \mapsto A_0 \\
& ijkhl \mapsto jiklh \mapsto jhlki \mapsto klhji \mapsto lkhij
\end{align*} \]

Let us consider $A_4$.

Let $\pi_1 : \mathbb{P}^1 \mapsto \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ at $A_4$. Let $U_4$ be the affine open set $\{X_4 \neq 0\}$ with coordinates $x = X_0/X_4$, $y = X_1/X_4$, $z = X_2/X_4$ and $t = X_3/X_4$. The polynomial defining $V \cap U_0$ is

\[ V \cap U_4 : f_6(x, y, z, t, 1) = a_{330000}x^3y^3 + \cdots + a_{002222}z^2t^2. \]

Locally the blow-up $\pi_1$ is given by the formulas:

\[
B_{x_1} : \begin{cases}
  x = x_1 \\
  y = x_1y_1 \\
  z = x_1z_1 \\
  t = x_1t_1
\end{cases} \quad \text{and} \quad B_{y_1} : \begin{cases}
  x = x_2y_2 \\
  y = y_2 \\
  z = y_2z_2 \\
  t = y_2t_2
\end{cases} \quad \text{and} \quad B_{z_1} : \begin{cases}
  x = x_3z_3 \\
  y = y_3z_3 \\
  z = z_3 \\
  t = z_3t_3
\end{cases} \quad \text{and} \quad B_{t_1} : \begin{cases}
  x = x_4t_4 \\
  y = y_4t_4 \\
  z = z_4t_4 \\
  t = t_4
\end{cases}
\]

and we consider, for example, $B_{x_1}$. The strict (or proper) transform $V_{x_1}$ of $V \cap U_4$ with respect to $B_{x_1}$ is given by

\[ V_{x_1} : \frac{1}{x_1^3} f_6(x_1, x_1y_1, x_1z_1, x_1t_1) = a_{330000}x_1^3y_1^3 + \cdots + a_{002222}x_1z_1^2t_1^2. \]

We impose on $V_{x_1}$ the double surface, better the double plane, $\{x_1 = y_1 = 0\}$ (i.e. we impose that such a plane is a locus of double points on $V_{x_1}$). Since the coefficients $a_{ijklh}$ are arbitrary, according to Bertini, this means that the plane is (at least) double on every monomial $a_{ijklh}x_4^m y_4^n z_4^p t_4^q$. It is consequently very easy to compute the conditions on the coefficients, in order that $V_{x_1}$ has the double plane $\{x_1 = y_1 = 0\}$.

Let us denote by $V_1$ the strict transform of $V$ with respect to the blow-up $\pi_1$. The above conditions on the coefficients impose on $V_1$ the double surface we called $S_4$, i.e. $S_4$ is a double surface on $V_1$ infinitely near $A_4 = (0, 0, 0, 0, 1)$ in its first neighbourhood.

Now, we impose a double surface $S_i$ infinitely near $A_i$, for $i = 0, 1, 2, 3$ by means of the above permutations without repeating the above calculations.
By imposing all the conditions, we obtain for $V$ an equation depending on 26 free coefficients (also called parameters) because some conditions are a duplicate of previous ones. Several of the 26 parameters can be chosen as equal to zero, because they are inessential in the computation of the birational invariants of a desingularization $\sigma_{|x}: X \rightarrow V$ of $V$, as well as in the computation of the dimensions of the images under pluricanonical transformations. The shortest form with the essential coefficients and defining our hypersurface with the above-said singularities is given by

$$f_6 = a_{31002}X_0^3X_1X_4^2 + a_{13020}X_0X_1X_3^2X_4^2 + a_{20301}X_0^2X_2X_4^3 + a_{20031}X_0^2X_3X_4^3$$

$$+ a_{02013}X_1^2X_3X_4^2 + a_{12210}X_0X_2X_4^2X_3 + a_{12120}X_0X_2X_4X_3^2$$

$$+ a_{11220}X_0X_1X_2X_4^2 + a_{11202}X_0X_1X_3X_4^2 + a_{10221}X_0X_2X_3X_4^3.$$

The hypersurface $V$, obtained for a generic choice of the parameters $a_{ijkl}$, will be called a generic $V$. In the sequel, when we shall consider our $V$, it is understood our generic $V$.

1.2 – Imposed and unimposed singularities on $V$.

We consider the hypersurface $V$ at the end of section 1.1.

Close to the singularities imposed on $V$ (the triple point $A_i$ having an infinitely near double surface $S_i$, $i = 0, 1, 2, 3, 4$), new singularities appear on the generic $V$, either actual or infinitely near. We call actual singularities the singularities on $V$ that are not infinitely near. Let us find the unimposed actual singularities on $V$ in the present section.

According to Bertini’s theorem (characteristic zero), the actual singularities on the generic $V$ belong to the base points of the linear system defining $V$. It is not difficult to find that the unimposed actual singularities are given by the following five double (straight) lines

$$\{X_0 = X_1 = X_i = 0\}, \ i = 2, 3, 4; \ \ \ \ \{X_j = X_3 = X_4 = 0\}, \ j = 0, 2;$$

and a double plane cubic

$$\{X_0 = a_{13020}X_1^2X_3 + a_{12210}X_1X_2^2 + a_{12120}X_1X_2X_3 + a_{11220}X_2X_3 = X_4 = 0\}.$$

In the following picture the five double lines are drawn in bold type.
In particular the unimposed actual singularities have codimension 2, therefore $V$ is reduced, irreducible and normal.

**Remark 1.1.** We note that the double singular curves, actual or infinitely near, do not give conditions of adjointness, that is, they do not give conditions to the hypersurfaces for them to be (any kind of) adjoints (cf. [S], Remark 17, section 11). In particular, such curves do not affect the birational invariant of a desingularization $X$ such as $q_1, q_2, p_1, p_2$, as well as the computation of $\text{dim } \varphi_{mK_X}(X)$. Moreover, we note explicitly that the singularity given by the double plane cubic can be resolved blowing up the plane containing it.

**Resolution of singularities of $V$**

The main purpose of this resolution of singularities of $V$ is to find the infinitely near unimposed singularities and to check that they do not give conditions to any kind of adjoints to $V$. More precisely we find that the infinitely near unimposed singularities are given by double singular lines and isolated double singular points.

1.3 – **Blowing up the triple point $A_4 \in U_4$.**

According to section 1.1, we consider the affine open set $U_4 = \{X_4 \neq 0\}$ of affine coordinates $(x, y, z, t)$. The actual singularities on $V$ belonging to $U_4$ are given by the two double lines

$$A_2A_4 \cap U_4, \quad A_3A_4 \cap U_4.$$  

The equation of $V \cap U_4$ is given by

$$f_6(x, y, z, t, 1) = a_{31002}x^3y + \cdots + a_{10221}xz^2t^2 = 0.$$
By using the notations of section 1.1, and denoting by $V_{x_1}, V_{y_2}, V_{z_3}, V_{t_4}$ the strict (or proper) transform of $V \cap U_4$ with respect to $\mathcal{B}_{x_1}, \mathcal{B}_{y_2}, \mathcal{B}_{z_3}, \mathcal{B}_{t_4}$, respectively, we obtain the following results

- $V_{x_1}$ is given by
  \[ V_{x_1} : \frac{1}{x_1^3} f_6(x_1, x_1 y_1, x_1 z_1, x_1 t_1) = a_{31002} x_1 y_1 + \cdots + a_{10221} x_1^2 z_1^2 t_1^2 = 0. \]

  On $V_{x_1}$ there is a unique singularity given by the double plane \( \{ x_1 = y_1 = 0 \} \); this means that the double plane is infinitely near $A_4$ in the first neighbourhood.

- $V_{y_2}$ is nonsingular.

- On $V_{z_3}$ there are two singularities: the double plane \( \{ y_3 = z_3 = 0 \} \) on the exceptional divisor $z_3 = 0$ and the double line \( \{ x_3 = y_3 = t_3 = 0 \} \) outside the exceptional divisor; it is the strict transform of the actual double line $A_2 A_4 \cap U_4$.

- On $V_{t_4}$ there are again two singularities: the double plane \( \{ y_4 = t_4 = 0 \} \) on the exceptional divisor $t_4 = 0$ and the double line \( \{ x_4 = y_4 = z_4 = 0 \} \) outside the exceptional divisor; it is the strict transform of the actual double line $A_3 A_4 \cap U_4$.

### 1.4 – The blow-up $\pi_2 : \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ of $\mathbb{P}_1$ along the surface $S_4$

#### 1.4.1 Let us consider $V_{x_1}$. On $V_{x_1}$ the surface $S_4$ is given by the double plane \( \{ x_1 = y_1 = 0 \} \).

Locally the blow-up along this plane is given by the formulas

\[
\mathcal{B}_{x_{11}} : \begin{cases}
x_1 = x_{11} \\ y_1 = x_{11} y_{11} \\ z_1 = z_{11} \\ t_1 = t_{11}
\end{cases} \quad \mathcal{B}_{y_{12}} : \begin{cases}
x_1 = x_{12} y_{12} \\ y_1 = y_{12} \\ z_1 = z_{12} \\ t_1 = t_{12}
\end{cases}.
\]

Let us denote by $V_{x_{11}}$, the strict transform of $V_{x_1}$ with respect to $\mathcal{B}_{x_{11}}$, and by $V_{y_{12}}$, the strict transform of $V_{x_1}$ with respect to $\mathcal{B}_{y_{12}}$.

- $V_{x_{11}}$ is nonsingular; its equation is given by
  \[ V_{x_{11}} : \frac{1}{x_{11}^3} f_6(x_{11}, x_{11}^2 y_{11}, x_{11} z_{11}, x_{11} t_{11}, 1) = a_{31002} y_{11} + \cdots + a_{10221} z_{11}^2 t_{11}^2 = 0. \]

- $V_{y_{12}}$ is nonsingular.
1.4.2 Let us consider $V_{z_3}$. On $V_{z_3}$ the surface $S_4$ is given by the double plane $\{ y_3 = z_3 = 0 \}$. Blowing up this plane, we find that there are no singularities infinitely near it.

1.4.3 Let us consider $V_{t_4}$. On $V_{t_4}$ the surface $S_4$ is given by the double plane $\{ y_4 = t_4 = 0 \}$. Blowing up this plane, again we find that there are no singularities infinitely near it.

1.5 – The blow-ups along the double lines that are strict transforms of the double lines $A_2A_4 \cap U_4$ and $A_3A_4 \cap U_4$.

Blowing up the strict transforms of $A_2A_4 \cap U_4$ and $A_3A_4 \cap U_4$, it is not difficult to see that infinitely near each of these double lines there is another double line and infinitely near the last double line there are no singularities.

The tree of the blow-ups resolving the singularities on $V \cap U_4$ is described below.

Where the nonsingular threefolds are drawn in bold type.

In the above sections 1.3–1.5, we gave, as an example, the blow-ups resolving the singularities of $V \cap U_4$ and we wrote the equations of the strict transforms that we need in the sequel. We calculated also the other similar desingularizations but we do not reproduce them here: we consider them as being achieved, as well as the complete desingularization of $V$. 
1.6 – The \(m\)-canonical adjoints to \(V \subset \mathbb{P}^4\).

Let

\[
P_r \overset{\pi_r}{\longrightarrow} \cdots \overset{\pi_3}{\longrightarrow} P_2 \overset{\pi_2}{\longrightarrow} P_1 \overset{\pi_1}{\longrightarrow} P_0 = \mathbb{P}^4
\]

be a sequence of blow-ups resolving the singularities of \(V\).

If we call \(V_i \subset P_i\); the strict transform of \(V_{i-1}\) with respect to \(\pi_i\), then we obtain from the above sequence

\[
V_r \overset{\pi'_r}{\longrightarrow} \cdots \overset{\pi'_3}{\longrightarrow} V_2 \overset{\pi'_2}{\longrightarrow} V_1 \overset{\pi'_1}{\longrightarrow} V_0 = V,
\]

where \(\pi'_i = \pi_i|_{V_i} : V_i \longrightarrow V_{i-1}\) and \(\sigma|_X : X \longrightarrow V\), \(\sigma = \pi_r \circ \cdots \circ \pi_1\), is a desingularization of \(V \subset \mathbb{P}^4\).

Let us assume that \(\pi_i\) is a blow-up along a subvariety \(Y_{i-1}\) of \(P_{i-1}\), of dimension \(j_{i-1}\), which can be either a singular or a nonsingular subvariety of \(V_{i-1} \subset P_{i-1}\) (i.e. \(Y_{i-1}\) is a locus of singular or simple points of \(V_{i-1}\)). Let \(m_{i-1}\) be the multiplicity of the variety \(Y_{i-1}\) on \(V_{i-1}\).

Let us set \(n_{i-1} = -3 + j_{i-1} + m_{i-1}\), for \(i = 1, \ldots, r\) and \(\deg(V) = d\).

A hypersurface \(\Phi_{m(d-5)}\) of degree \(m(d-5)\) in \(\mathbb{P}^4\) is an \(m\)-canonical adjoint to \(V\) (with respect to the sequence of blow-ups \(\pi_1, \ldots, \pi_r\)) if the restriction to \(X\) of the divisor

\[
D_m = \pi^*_r \{ \pi^*_r \{ \cdots \{ \pi^*_3 (\Phi_{m(d-5)}) \cdots \} - mn_0 E_1 \cdots \} - mn_{r-2} E_{r-1} \} - mn_{r-1} E_r
\]

is effective, i.e. \(D_m|_X \geq 0\), where \(E_i = \pi^{-1}(Y_{i-1})\) is the exceptional divisor of \(\pi_i\) and \(\pi^*_i : Div(P_{i-1}) \longrightarrow Div(P_i)\) is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S1], sections 1.2).

An \(m\)-canonical adjoint \(\Phi_{m(d-5)}\) is a global \(m\)-canonical adjoint to \(V\) (with respect to \(\pi_1, \ldots, \pi_r\)) if the divisor \(D_m\) is effective on \(P_r\), i.e. \(D_m \geq 0\) (loc. cit.).

Note that, if \(\Phi_{m(d-5)}\) is an \(m\)-canonical adjoint to \(V\), then \(D_m|_X \equiv mK\), where ‘\(\equiv\)’ denotes linear equivalence and \(K\) denotes a canonical divisor on \(X\).

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that the blow-up \(\pi_1\) is the blow-up at the 3-ple point \(A_4\), \(\pi_2\) is the blow-up along the double surface \(S_4\) infinitely near \(A_4\) (see also section 1.1, 1.3 and 1.4), \(\pi_3\) is the blow-up at the triple point \(A_3\), \(\pi_4\) is the blow-up along the double surface \(S_3\) infinitely near \(A_3\), \(\pi_5\) is the blow-up at the triple point \(A_2\), \(\pi_6\) is the blow-up along the double surface \(S_2\) infinitely near \(A_2\), \(\pi_7\) is the blow-up at the triple point \(A_1\), \(\pi_8\) is the blow-up along the double surface \(S_1\) infinitely near \(A_1\), and \(\pi_9\) is the blow-up at the triple point \(A_0\), \(\pi_{10}\) is the blow-up along the double surface \(S_0\) infinitely near \(A_0\).
The hypersurface $V$ has degree $d = 6$ and $D_m$ is given by:
\[
\hat{D}_m = \pi^*_r \cdots \{ \pi^*_2 [ \pi^*_1 (\Phi_m) - mE_2] - mE_4 \} - mE_6 - mE_8 - mE_{10} + \sum mE,
\]
where $E_i$ is the exceptional divisor of the blow-up $\pi_i$ and, to be more specific, $E_2$ is the exceptional divisor of the blow-up $\pi_2$ along $S_4, \ldots, E_{10}$ is the exceptional divisor of the blow-up $\pi_{10}$ along $S_0$.

No other exceptional divisors are subtracted in $D_m$ because, as we said, the unimposed singularities are either actual or infinitely near double singular curves or isolated double points on our (generic) $V$. We note that the exceptional divisors of the blow-ups at double isolated points appear with coefficient $n_h = -1$, we have indicated these divisors as $\sum mE$. From here on, we omit writing $\sum mE$, because they are not essential in the computation of the birational invariants, as well as in the computation of $\dim \varphi_{|mK_X|}(X)$, that we shall consider.

1.7 – The global and non-global m-canonical adjoints to $V \subset \mathbb{P}^4$.

**Proposition 1.** If we consider a non-global $m$-canonical adjoint to $V$

\[
\Phi_m : F_m(X_0, X_1, X_2, X_3, X_4) = \sum_{i+j+k+l=m} b_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l = 0,
\]

where $b_{ijkl} \in k$, then a form $A = A(X_0, X_1, X_2, X_3, X_4)$ exists such that $\Phi_m^* = F_m - A \delta = 0$ is a global $m$-canonical adjoint to $V$. In other words, the following equality holds

\[
\Phi_m|_V = \Phi_m^*|_V.
\]

Proposition 1 holds for the three constructions in the present paper (see Proposition 2 in the Appendix). The three proofs are similar, so, to avoid unnecessary repetitions, we produce only one proof in the Appendix. There are two ways to prove these Propositions: the first way is contained in [S2], cf. the proof of Lemma 5, section 18, p. 1177; the second way is due to Maria Cristina Ronconi. We reproduce in the Appendix the proof of Mrs. Ronconi.

**Lemma 1.** The global $m$-canonical adjoints to $V$ are given by

\[
\Psi_m : \sum_{i+j+k+l=m} c_{ijkl} X_0^i X_1^j X_2^k X_3^l X_4^l = 0,
\]
where \( c_{ijkl} \in k \) and where \( l \geq i \geq h \geq j \geq l, \ i \geq k, \) i.e. \( l = i = h = j \) and \( i \geq k \) (for all monomials).

**Proof of Lemma 1.** Let us consider a global \( m \)-canonical adjoint to \( V \)

\[
\Psi_m : \sum_{i+j+k+l=m} c_{ijkl} x_0^i x_1^j x_2^k x_3^l x_4^l = 0.
\]

The total transform \( \Psi^* \) of \( \Psi_m \cap U_4 \) with respect to \( B_{x_1} \) (section 1.1) is given by

\[
\Psi^* = B^*_{x_1} (\Psi_m \cap U_4) : \sum_{i+j+k+l=m} c_{ijkl}(x_1)^i(y_1)^j(z_1)^k(t_1)^l = 0,
\]

The double surface \( S_4 \) infinitely near \( A_4 \) in affine coordinates \((x_1, y_1, z_1, t_1)\) is given by \{\( x_1 = y_1 = 0 \)\} and the blow-up \( \pi_2 \) along \( S_4 \) is given locally by the formulas \( B_{x_1} \) and \( B_{y_2} \) (see section 1.4.1).

The total transform \( \Psi^{**} \) of \( \Psi^* = B^*_{x_1} (\Psi_m \cap U_4) \) with respect to \( B_{x_1} \) is given by

\[
\Psi^{**} = B^*_{x_11} (\Psi^*) : \sum_{i+j+k+l+m} c_{ijkl}(x_{11})^i(y_{11})^j(z_{11})^k(t_{11})^l = 0.
\]

Since \( \Psi_m \) is a global \( m \)-canonical adjoint to \( V \), by definition in (\(^\diamond\)), section 1.6, we have \( D_m \geq 0 \).

We note that \( B_{x_11} \circ B_{x_1} \) coincides, up to isomorphisms, with the desingularization \( \sigma_x \) on the affine open set \( V_{x_11} \). In fact, \( V_{x_11} \) is nonsingular (see the tree of blow-ups, section 1.5) and then it is isomorphic to a Zariski open set on the desingularization \( X \) of \( V \). The above coincidence and the inequality \( D_m \geq 0 \) imply the inequality \( \pi_2^*[\sigma_x^*(F_m) - mE_2] \geq 0 \) and the following inequality between divisors (of rational functions) on the affine open set \( U_{x_11} \) of (affine) coordinates \((x_{11}, y_{11}, z_{11}, t_{11})\)

\[
\left( \frac{\Psi^{**}}{x_{11}^m} \right) = \left( \frac{1}{x_{11}^m} \right) \left( \sum_{i+j+k+l+m} c_{ijkl}(x_{11})^{i+2j+k+h}y_{11}^jz_{11}^k t_{11}^l = 0 \right) \geq 0.
\]

Since, on the affine open set \( U_{x_{11}}, x_{11} = 0 \) is the local equation, of the exceptional divisor \( E_2 \) of the blow-up \( \pi_2 \), this last inequality is equivalent to

\[ i + 2j + k + h - m \geq 0, \ \text{i.e.} \ j \geq l. \]

We have proved the above inequality for a particular sequence of open sets.
appearing in the blow-ups. If we consider all the other sequences of open sets, we find the same inequality.

This proves the last inequality in the sequence of inequalities 
\( l \geq i \geq h \geq j \geq l \) in the statement of Lemma 1.

If we consider the singular points \( A_3, A_2, A_1 \) and \( A_0 \) and we go on with the blow-ups \( \pi_3, \ldots, \pi_{10} \), then all the other inequalities follow in a similar way. More precisely, if we consider the point \( A_3 \), then we obtain \( i \geq h \); considering \( A_2 \) we obtain \( i \geq k \); \( A_1 \) leads to \( h \geq j \) and \( A_0 \) to \( l \geq i \).

We note that, up to isomorphisms, we can start by blowing up \( A_i \) first, with \( i \neq 4 \), and in this case, we repeat for \( A_i \) what we did for \( A_4 \) obtaining in the same way all the inequalities.

So, Lemma 1 is proved.

1.8 – Computing the plurigenera of \( X \).

Now, we consider [S1], Corollary 8, section 3: if \( V \) is normal, there is an isomorphism of projective spaces for any \( m \geq 1 \)

\[
\left( \text{linear system of} \quad m - \text{canonical adjoints to } V \right) \left|_V \right. \rightarrow |mK_X| \\
\Phi_m \left|_V \right. \rightarrow D_m|_X.
\]

\( D_m \) is defined in (\( ^{\diamond} \)), section 1.6.

Bearing in mind that our purpose is to compute the \( m \)-canonical genus \( P_m = \dim |mK_X| + 1 = \dim (\text{linear system of } m - \text{canonical adjoints}) \left|_V \right. + 1 \), we can substitute \( \Phi_m \) with \( \Phi'_m \) if \( \Phi'_m \left|_V \right. = \Phi_m \left|_V \right. \).

Next, the Proposition 1 in section 1.7 tells us that in order to compute the \( m \)-genus \( P_m \), we can restrict ourselves to consider global \( m \)-canonical adjoints to \( V \) and Lemma 1 in the same section tells us that the global \( m \)-canonical adjoints are given by

\[
\Psi_m : \sum_{4s+v=m} c_{4s+v}X_0^sX_1^vX_2^vX_3^vX_4^v = 0,
\]

where \( c_{4s+v} \in k \) and \( v \leq s, s > 0 \).

By doing the easy calculations, we obtain

\[
p_9 = P_1 = P_2 = P_3 = 0; \\
P_4 = 1, \text{ the global 4-canonical adjoint is defined by } X_0X_1X_2X_4; \\
P_5 = 1, \text{ the global 5-canonical adjoint is defined by } X_0X_1X_2X_3X_4;
\]
$P_6 = P_7 = 0$;
$P_8 = 1$, the global 8-canonical adjoint is defined by $X^2_0X^2_1X^2_2X^2_3$;
$P_9 = 1$, the global 9-canonical adjoint is defined by $X_0^3X_1^3X_2^3X^3_3$;
$P_{10} = 1$, the global 10-canonical adjoint is defined by $X^2_0X^2_1X^2_2X^2_3X^2_4$;
$P_{11} = 0$;
$P_{12} = P_{13} = P_{14} = P_{15} = P_{16} = P_{17} = P_{18} = P_{19} = 1$;
$P_{20} = 2$ the global 20-canonical adjoints are defined by $\lambda_1X_0^5X_1^5X_3^5X_4^5 + \lambda_2X_0X_1X_2X_3X_4 = X_0^4X_1^4X_2^4X_3^4X_4(\lambda_1X_0X_1X_3X_4 + \lambda_2X_2^4)$, $\lambda_i \in k$; and so on.

**Remark 1.2.** 1) We have seen that the first integer $m$ such that $P_m = 2$ is $m = 20$ and the 20-canonical adjoints are given, up to fixed components, by the pencil $\lambda_1X_2^4 + \lambda_2X_0X_1X_3X_4 = 0$. Continuing the above list, we obtain

2) the minimum integer $m_0$ such that $P_m \geq 2$ for $m \geq m_0$ is given by $m_0 = 32$;

3) the first integer $m$ such that $P_m = 3$ is $m = 40$ and the global 40-canonical adjoints are defined by $\mu_1X_0^8X_1^8X_2^8X_3^8X_4^8 + \mu_2X_0X_1X_2X_3X_4 + \mu_3X_0X_1X^4_2X^4_3X^4_4 = X_0X_1X_3X_4(\mu_1X_2^8 + \mu_2X_0X_1X_2X_3X_4 + \mu_3X_0X_1X^4_2X^4_3X^4_4)$, $\mu_i \in k$;

1.9 – The $m$-canonical transformation $\varphi_{|mK_X|}$.

Let us consider the following triangle

\[ \begin{array}{ccc}
X & \xrightarrow{\varphi_{|mK_X|}} & P_m^{-1} \\
\sigma_{|X} & \downarrow & \varphi_{L_m} \\
\downarrow & & \\
V & & 
\end{array} \]

where $\sigma_{|X} : X \longrightarrow V$, with $\sigma = \pi_r \circ \cdots \circ \pi_1$, denotes our desingularization of $V$, where $L_m$ denotes the (incomplete) linear system of $m$-canonical adjoints to $V$ restricted to $V$ and $\varphi_{L_m}$ the rational transformation defined by the linear system $L_m$.

The above triangle is commutative. This follows from the fact that the divisors of the linear system $|mK_X|$ on $X$ are the divisors $D_{m|X}$, where $D_m = \pi_m^*(\pi_m^{-1}([\cdots \pi_1^*(\Phi_m) - mn_0E_1 - \cdots] - mn_{r-2}E_{r-1} - mn_{r-1}E_r)$, with $\Phi_m$ varying in the linear system of $m$-canonical adjoints to $V$. 

In the present section, we want to find the minimum integer $m_0$ such
that the $m$-canonical transformation $\varphi_{[mK_X]}$ enjoys the property: $\dim\varphi_{[mK_X]}(X) = 1$, for $m \geq m_0$. The values of $m$ such that $\dim\varphi_{[mK_X]}(X) = 1$ are exactly given by the values for which $P_m \geq 2$. This is a consequence of the following facts:

a) restricting the global $m$-canonical adjoints to $V$ there is no identifications;
b) the commutativity of the above triangle;
c) 4), 5) in Remark 3.

So, thanks to 2) in Remark 3, the above value of $m_0$ is given by $m_0 = 32$.

Therefore, the Kodaira dimension of $X$ is $\kappa(X) = 1$ and the minimum integer $m_0$ such that $\dim\varphi_{[mK_X]}(X) = 1$, for $m \geq m_0$, is given by $m_0 = 32$.

Moreover, we note that the generic fiber of the rational transformation $\varphi_{[mK_X]}$ is irreducible (by Bézout theorem) for those values of $m$ for which $P_m = 2$, whereas such generic fiber is reducible for those values of $m$ for which $P_m \geq 3$.

1.10 – Computing the irregularities of $X$.

It remains to prove that $q_i = \dim_k H^i(X, O_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(X, O_X) = q(S_r) = \dim_k H^1(S_r, O_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section $S$ of $V$ (cf. [S1], section 4, for instance). $S$ has several isolated (actual or infinitely near) double points and no other singularities. This follows from the fact that the hypersurface $V$, outside the points $A_0, A_1, A_2, A_3$ and $A_4$, only has actual or infinitely near double curves or isolated double points. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36), section 4 in [S1], which states that 

$$q_2 = p_g(X) + p_g(S_r) - \dim_k(W_2),$$

where $W_2$ is the vector space of the degree 2 forms defining global adjoints $\Phi_2$ to $V$, i.e. defining hyperquadrics $\Phi_2$ such that 

$$\pi_r^* \cdots \pi_r^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_{10} \geq 0,$$

(cf. the expression of $D_m$ in (5), section 1.6). So the above hyperquadrics $\Phi_2$ are those passing through the points $A_0, A_1, A_2, A_3$ and $A_4$. Thus, we
have \( \dim_k(W_2) = 15 - 5 = 10 \). It follows from \( p_g(S_v) = 10 \) and \( p_g(X) = 0 \) (cf. section 1.8), that \( q_2 = 0 \).

2. Construction of \( X' \) and of \( X'' \).

In this chapter we construct two threefolds \( X' \) and \( X'' \), with the properties described in the Introduction, as desingularizations of two hypersurfaces \( V' \) and \( V'' \) of degree six in \( \mathbb{P}^4 \). Using the same method of Chapter 1, we impose sextic hypersurfaces to have triple points at the coordinate points \( A_\lambda \), \( \lambda = 0, \ldots, 4 \), each one with infinitely near a double plane, namely a double surface whose local equations are linear, obtained with slight modifications of the permutations described in Chapter 1, § 1.1.

Indeed, the singularities of \( V', V'' \) are of the same type as the singularities of \( V \) in Chapter 1. The difference with \( V \) (and between \( V' \) and \( V'' \)) is mainly given by the position of the double surface infinitely near to \( A_4 \), which will imply the difference in the birational equivalence classes of \( V, V' \) and \( V'' \).

The explicit equation of \( V' \) is
\[
f_6' = a_{30102}X_0^3X_2X_4^2 + a_{13020}X_0X_1X_2^2X_3^2 + a_{20301}X_0^2X_2^2X_3^2X_4 + a_{20031}X_0^2X_2^3X_3X_4 + a_{10203}X_0X_2^3X_3^2X_4 + a_{12210}X_0X_1X_2^2X_3^2X_4 + a_{11220}X_0X_1X_2^2X_3X_4^2 + a_{02112}X_1^2X_2X_3X_4^2 = 0,
\]
while the equation of \( V'' \) is
\[
f_6'' = a_{30012}X_0^3X_3X_4^2 + a_{13020}X_0X_1X_3^2X_4 + a_{20301}X_0^2X_3^2X_4 + a_{20031}X_0^2X_3^3X_4 + a_{10023}X_0X_3^2X_4^2 + a_{12201}X_0X_1X_3^2X_4 + a_{11202}X_0X_1X_3X_4^2 + a_{10221}X_0X_2X_3^2X_4 + a_{10212}X_0X_2X_3X_4^2 + a_{02112}X_2X_3X_4^2 = 0,
\]
where the coefficients \( a_{ijkl} \) are sufficiently general. (Actually, one can construct such a threefold \( X'' \) from a hypersurface \( V'' \) depending on 28 — instead of 12 — parameters, but we used only 12 of them for brevity.)

Reasoning like in Chapter 1, one sees that \( V', V'' \) are normal, they have triple points at \( A_\lambda \), \( 0 \leq \lambda \leq 4 \), they are double along the lines \( A_0A_1, A_1A_2, A_1A_4, A_2A_3 \) and along the rational cubic plane curve:
\[
X_0 = X_4 = a_{13020}X_1X_3 + a_{12210}X_1X_2^2 + a_{11220}X_2^2X_3 = 0.
\]
The further actual singularity of \( V' \) is the double line \( A_3A_4 \), while the
further actual singularities of $V''$ are the double line $A_2A_4$ and two double rational cubic plane curves:

\[
X_0 = X_1 = a_{1022}X_2^2X_3 + a_{10212}X_2^2X_4 + a_{10023}X_3X_4^2 = 0, \\
X_2 = X_3 = a_{30012}X_2^2X_4 + a_{22011}X_0X_4^2 + a_{12012}X_1^2X_4 = 0.
\]

Setting $P_0 = P^4$, we perform the blow-ups $\pi_i : P_i \rightarrow P_{i-1}, i = 1, \ldots, 9$, where $\pi_{2i+1}, \lambda = 0, \ldots, 4$, is the blow-up at $A_i \in U_\lambda = \{X_\lambda \neq 0\}$ and $\pi_{2i+2}$, $\lambda = 0, \ldots, 3$, is the blow-up along the double surface $S_\lambda$ infinitely near to $A_i$, which is the same for $V$, $V'$ and $V''$.

With the usual affine coordinates $x, y, z, t \in U_\lambda, \lambda = 0, \ldots, 4$, each blow up $\pi_{2i+1}$ is given locally by the formulas $B_{2i}, B_{2i+1}, B_{3i}$, written in § 1.1. With respect to $B_{2i}$, we see that the local equation of the double surface $S_0$, infinitely near to $A_0$, is $y_2 = t_2 = 0$; the local equation of $S_1$ is $y_2 = z_2 = 0$; those of $S_2$ and of $S_3$ are $x_2 = y_2 = 0$.

Finally, for $V'$, let $\pi'_{10} : P'_{10} \rightarrow P_9$ be the blow up along the surface $S'_4$ infinitely near to $A_4$, with local equation $y_2 = z_2 = 0$ with respect to $B_{2i}$. For $V''$, let $\pi''_{10} : P''_{10} \rightarrow P_9$ be the blow up along the surface $S''_4$ infinitely near to $A_4$, with local equation $y_2 = t_2 = 0$ with respect to $B_{2i}$.

We then checked that the strict transforms of $V'$ in $P'_{10}$ and of $V''$ in $P''_{10}$ are singular along double curves only, and that no further essential singularity appears in the resolution process. In other words, the double surfaces infinitely near to the coordinate triple points are the only essential singularities of $V'$ and of $V''$. Therefore we may compute global $m$-canonical adjoints to $V'$ and to $V''$ in the same way we did in Chapter 1.

**Lemma 2.** Let $F = \sum b_{ijkl}X_i^1X_j^1X_k^1X_l^1$ be a homogeneous polynomial of degree $m$, i.e. with $i + j + k + l = m$ and $b_{ijkl} \in k$. Then $\Phi_m = \{F = 0\} \subset P^4$ is a global $m$-canonical adjoint to $V'$ [resp. to $V''$] if and only if $j \leq h \leq i = k = l$ [resp. $j \leq i = h = l \geq k$] for each monomial in $F$.

**Proof.** Following the proof of Lemma 1 in Chapter 1, we compute the conditions imposed by the double surfaces infinitely near to $A_0 [A_1, A_2, A_3$, resp.] and we find out that $l \geq i [h \geq j, i \geq k, i \geq h$, resp.]. Concerning $A_4$, we find that $k \geq l$ for $V''$ and that $h \geq l$ for $V''$. It follows that $i = k = l$ for $V'$ and that $i = h = l$ for $V''$, which conclude the proof.

By Proposition 2 in the Appendix, global $m$-canonical adjoints to $V'$ and to $V''$ are enough to compute the plurigenera $P_m(X'), P_m(X'')$ of $X'$ and $X''$, and their pluricanonical transformations, which we study now.
2.1 – Canonical adjoints to $V'$, pluricanonical transformations of $X'$.

By Lemma 2, the global $m$-canonical adjoints to $V'$ are given by

$$\Phi_m : \sum_{5j+4u+3v = m, j \geq 0, u \geq j, v \geq k} b_{juk} X_1^j X_2^k (X_0 X_2 X_4)^{j+u+v} = 0,$$

where $b_{juk} \in k$. Denote by $A'_m$ the $k$-vector space of the polynomials defining global $m$-canonical adjoints to $V'$. Setting $Y = X_0 X_2 X_4$, it follows that

$$A'_1 = A'_2 = \{0\}, \quad A'_3 = \langle X_3 Y \rangle, \quad A'_4 = \langle X_3 Y \rangle,$$

$$A'_5 = \langle X_1 X_3 Y \rangle, \quad A'_6 = \langle Y^2 \rangle, \quad A'_7 = \langle X_3 Y^2 \rangle,$$

$$A'_8 = \langle X_1 X_3 X_3 Y^2 \rangle, \quad A'_9 = \langle X_1 X_3^2 Y \rangle, \quad A'_{10} = \langle X_1 X_3 Y \rangle X_3 Y^2,$$

$$A'_{11} = \langle X_1 X_3 X_3 Y^3 \rangle, \quad A'_{12} = \langle X_1 X_3^2 X_3 Y \rangle Y^3,$$

$$A'_{13} = \langle X_1 X_3 X_3 X_3 X_3 Y \rangle X_3 Y^3, \quad A'_{14} = \langle X_1 X_3^2 X_3 X_1 X_3 Y \rangle X_3 Y^3,$$

and so on, which give the $P_m(X')$'s written in Table 1 in the Introduction.

We now study the $m$-canonical transformations $\varphi'_m = \varphi_{[mK_X]}$ of $X'$. We will show that $X'$ has Kodaira dimension 2 and that $\varphi'_m(X')$ has dimension 2 if and only if $m \geq 12$. Clearly, $\dim \varphi'_m(X') < 2$ if $m < 12$.

Then, it is easy to check that $\varphi'_m(X') = \Gamma^2$ for $m = 12, 13, 14$. Since $P_3(X') = 1$, it follows that $\dim \varphi'_m(X') \geq 2$ also for each $m \geq 15$.

An upper bound to $P_m(X')$ is given by the function $v: \mathbb{N} \to \mathbb{N}$,

$$v(m) = \#\{(j, u, v) \in \mathbb{N}^3 \mid 5j + 4u + 3v = m\} \geq P_m(X'),$$

where $\mathbb{N}$ is the set of non-negative integers. We then see that

$$P_m(X') \leq v(m) \leq \left(\frac{m}{3} + 1\right)\left(\frac{m}{4} + 1\right),$$

thus the Kodaira dimension of $X'$ is 2 and hence $\dim \varphi'_m(X') = 2$ for $m \geq 12$.

2.2 – Canonical adjoints to $V''$, pluricanonical transformations of $X''$.

By Lemma 2, the global $m$-canonical adjoints to $V''$ are given by

$$\Phi_m : \sum_{3i+j+k = m, i \geq 0, j \geq i, k \geq 0} b_{ijk} X_1^i X_2^j (X_0 X_3 X_4)^i = 0,$$

where $b_{ijk} \in k$. Denote by $A''_m$ the $k$-vector space of the polynomials defining global $m$-canonical adjoints of $X''$. Setting $Y = X_0 X_3 X_4$, it follows
that
\[ A_i' = A''_i = \{0\}, \quad A''_3 = \langle Y \rangle, \quad A''_4 = \langle X_1, X_2 \rangle Y, \]
\[ A''_5 = \langle X_1 X_2 Y \rangle, \quad A''_6 = \langle Y^2 \rangle, \quad A''_7 = \langle X_1, X_2 \rangle Y^2, \]
\[ A''_8 = \langle X_1^2 X_2, X_1 X_2^2 \rangle Y^2, \quad A''_9 = \langle X_1^2 X_2, X_1 X_2^2, Y \rangle Y^2, \]
\[ A''_{10} = \langle X_1^2 X_2, X_1 Y, X_2 Y \rangle Y^2, \quad A''_{11} = \langle X_1^2, X_1 X_2, X_2^2 \rangle Y^3, \]
\[ A''_{12} = \langle X_1^3, X_1^2 X_2, X_1 X_2^2, X_2^3, Y \rangle Y^3, \]
\[ A''_{13} = \langle X_1^3 X_2, X_1^2 X_2^2, X_1 X_2^3, X_1 Y, X_2 Y \rangle Y^3, \]
\[ A''_{14} = \langle X_1^3 X_2, X_1^2 X_2^2, X_1 X_2^3, X_2 Y, X_1 X_2 Y, X_2^2 Y \rangle Y^3, \]

and so on, which give the \( P_m(X'') \)'s written in Table 1 in the Introduction.

We next describe the \( m \)-canonical transformations \( \varphi''_m = \varphi_{|mK_{X''}|} \) of \( X'' \).

We claim that \( X'' \) has Kodaira dimension 2 and that \( \varphi''_m(X'') \) has dimension 2 if and only if \( m = 9, 10, \) or \( m \geq 12. \) Clearly, \( \dim \varphi''_m(X'') < 2 \) if \( m < 8. \)

Setting \( W_0 = X_1^2 Y^2, \ W_1 = X_1 X_2 Y^2 \) and \( W_2 = X_2^2 Y^2, \) we see that \( \varphi''_8(X'') \) is the plane conic \( W_2^2 = W_0 W_2 \) in \( \mathbb{P}^2 \) with coordinates \( W_0, W_1, W_2. \) One sees that \( \varphi''_{11}(X'') \) is a plane conic too, hence \( \dim \varphi''_8(X'') = \dim \varphi''_{11}(X'') = 1. \)

Moreover, we easily see that \( \varphi''_9(X'') = \varphi''_{10}(X'') = \mathbb{P}^2 \) and that \( \varphi''_{12}(X'') \) is a surface scroll in \( \mathbb{P}^4, \) namely a cone over a rational normal curve in \( \mathbb{P}^3. \) In coordinates \( Z_0 = X_1^3 Y^3, \ Z_1 = X_1^2 X_2 Y^3, \ Z_2 = X_1 X_2^2 Y^3, \ Z_3 = X_2^3 Y^3, \ Z_4 = Y^4, \) the equations of \( \varphi''_{12}(X'') \) are indeed
\[ \varphi_{12}(X'') : \text{rank} \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix} = 1. \]

One similarly sees that \( \varphi''_{13}(X'') \) and \( \varphi''_{14}(X'') \) are surface scrolls in \( \mathbb{P}^4, \) hence \( \dim \varphi''_m(X'') = 2 \) if \( m = 9, 10, 12, 13, \) or 14. Since \( P_3 = 1, \) it follows that \( \dim \varphi''_m(X'') \geq 2 \) for each \( m \geq 15. \)

An upper bound to \( P_m(X'') \) is given by the function \( \mu : \mathbb{N} \to \mathbb{N}, \)
\[ \mu(m) = \# \{(i, j, k) \in \mathbb{N}^3 \mid 3i + j + k = m \} \geq P_m(X''), \]
and, setting \( m = 3n + \varepsilon, \) with \( \varepsilon \in \{0, 1, 2\}, \) we see that
\[ P_m(X'') \leq \mu(m) = \sum_{i=0}^{n} \left( 3(n - i) + 1 + \varepsilon \right) = (n + 1) \left( \frac{3}{2} n + 1 + \varepsilon \right). \]

Thus the Kodaira dimension of \( X'' \) is 2 and \( \dim \varphi''_m(X'') = 2 \) for \( m \geq 12. \)

Finally, the same proof as in the case of the threefold \( X, \) cf. § 1.10, shows that the irregularities of \( X' \) and that of \( X'' \) are \( q_1 = q_2 = 0. \) This concludes the proof that \( X' \) and \( X'' \) have the properties described in the Introduction.
Appendix. Non-global canonical adjoints can be made global.

In this appendix we show that, in order to compute \( m \)-canonical adjoints to \( V, V' \) and \( V'' \), it suffices to compute global \( m \)-canonical adjoints.

We point out that the key ideas of these proofs are due to Maria Cristina Ronconi (cf. her approach in [Ro], § 4).

**Proposition 2.** Let \( F = \sum_{i+j+k+h+l=m} b_{ijkl} X_0^i X_1^j X_2^k X_3^h X_4^l \) be a homogeneous polynomial of degree \( m \), with \( b_{ijkl} \in k \), defining a non-global \( m \)-canonical adjoint to \( V [V', V'', \text{resp.}] \). Then there is a homogeneous polynomial \( A(X_0, \ldots, X_4) \) of degree \( m - 6 \) such that \( F - A f_0 \) \( F - A f_0', \ F - A f_0'' \), \( \text{resp.} \) defines a global \( m \)-canonical adjoint to \( V [V', V'', \text{resp.}] \).

We first write the proof of Proposition 2 for \( V \). We will then show what to change for \( V' \) and \( V'' \). We begin with some definitions and a lemma.

Let us define 5 functions \( \sigma_z : \mathbb{N}^5 \rightarrow \mathbb{N} \) (where \( \mathbb{N} \) is the set of non-negative integers), \( 0 \leq \lambda \leq 4 \), as follows: for each \( z = (i, j, k, h, l) \in \mathbb{N}^5 \), we set

\[
\begin{align*}
\sigma_0(z) &= |z| + l - i, \\
\sigma_1(z) &= |z| + h - j, \\
\sigma_2(z) &= |z| + i - k, \\
\sigma_3(z) &= |z| + i - h, \\
\sigma_4(z) &= |z| + j - l,
\end{align*}
\]

where \( |z| = i + j + k + h + l \). For brevity, we write \( X^z = X_0^i X_1^j X_2^k X_3^h X_4^l \).

The function \( \sigma_z \) will help to understand what happens to a monomial \( X^z \) appearing in the equation of a canonical adjoint when blowing up the point \( A_z \) and the surface \( S_z \) infinitely near to \( A_z \). Roughly speaking, in our situation the equation of an exceptional divisor corresponding to \( S_z \) is given by just a coordinate variable and \( \sigma_z \) counts how many times that variable appears in a monomial (cf. the proof of Lemma 1 in Chapter 1).

Fix \( \lambda, 0 \leq \lambda \leq 4 \). For any homogeneous polynomial \( G = \sum_{|z| = m} c_z X^z \in K[X] = K[X_0, \ldots, X_4] \) of degree \( m > 0 \), we define the integer

\[
r_{\lambda}(G) = \min \{ \sigma_z(z) : c_z \neq 0 \},
\]

and the polynomial

\[
G^{(\lambda)} = \sum_{z : \sigma_z(z) = r_{\lambda}(G)} c_z X^z,
\]

which is the part of \( G \) with monomials \( X^z \) such that \( \sigma_z(z) = r_{\lambda}(G) \). The de-
finitition implies that, if \( G^{(i)} \neq G \), then

\[
\rho_\lambda(G - G^{(i)}) \geq \rho_\lambda(G) + 1.
\]

Roughly speaking, \( \rho_\lambda(G) \) counts how many times the variable defining the exceptional divisor corresponding to \( S_\lambda \) factorizes all monomials of \( G \).

For each \( 0 \leq \lambda \leq 4 \), we see that \( \rho_\lambda(f_0) = 5 \) and we set \( p_\lambda = f_6^{(i)} \), i.e.

\[
p_0 = a_{31002}X_0^3X_1X_2^2 + a_{13020}X_0X_1X_2^3 + a_{20301}X_0^2X_1^3X_2 + a_{20031}X_0^3X_2^3X_4
+ a_{12210}X_0X_1^2X_2X_3 + a_{12120}X_0X_1^2X_2^2X_3 + a_{11220}X_0X_1X_2^2X_3^2,
\]

\[
p_1 = a_{31002}X_0^3X_1X_2^2 + a_{13020}X_0X_1X_2^3 + a_{02013}X_0^2X_1X_2^2X_3 + a_{12210}X_0X_1X_2^2X_3
\]

\[
p_2 = a_{20301}X_0^2X_1^2X_2 + a_{12210}X_0X_1X_2^2X_3 + a_{11220}X_0X_1X_2^2X_3 + a_{10221}X_0X_2X_3^2X_4,
\]

\[
p_3 = a_{13020}X_0X_1X_2^3 + a_{20031}X_0^2X_1^3X_2 + a_{02013}X_0^2X_1^2X_2 + a_{12120}X_0X_1X_2^2X_3
+ a_{11220}X_0X_1X_2^2X_3 + a_{11022}X_0X_1X_2^2X_3 + a_{10221}X_0X_2X_3^2X_4.
\]

\[
p_4 = a_{31002}X_0^3X_1X_2^2 + a_{20301}X_0^2X_1^2X_2 + a_{20031}X_0^2X_1^2X_2 + a_{20031}X_0X_1X_2^3X_4
+ a_{02013}X_0^2X_1X_2^3X_4 + a_{11022}X_0X_1X_2^2X_3 + a_{10221}X_0X_2X_3^2X_4.
\]

**Lemma 3.** Fix \( \lambda \), \( 0 \leq \lambda \leq 4 \). Assume that the homogeneous polynomial \( G = \sum_{|z| = m} c_zX^z \) of degree \( m \) defines an \( m \)-canonical adjoint to \( V \). If \( \rho_\lambda(G) < m \), then there exists a homogeneous polynomial \( B_\lambda \) of degree \( m - 6 \) such that \( G^{(i)} = B_\lambda p_\lambda \) and \( B_\lambda^{(i)} = B_\lambda \). Moreover, we have \( \rho_\lambda(B_\lambda) = \rho_\lambda(G) - 5 \) and \( \rho_\lambda(G - f_0) \geq \rho_\lambda(G) + 1 \).

**Proof.** We show in detail the case \( \lambda = 4 \), namely we see what happens when we blow up \( A_4 \in U_4 = \{X_4 \neq 0\} \) and the double surface \( S_4 \) infinitely near to \( A_4 \). We leave the other similar cases \( 0 \leq \lambda \leq 3 \) to the reader.

On \( U_4 \) with coordinates \( x, y, z, t \), we consider the composition \( B_{x_1} \circ B_{x_1} \) (cf. the proof of Lemma 1 in Chapter 1). We set \( \xi = (x_{11}, y_{11}, z_{11}, t_{11}) \) and \( \nu = (x_{11}, y_{11}, z_{11}, x_{11}t_{11}, 1) \).

Since \( G = \sum_{|z| = m} c_zX^z \) defines an \( m \)-canonical adjoint to \( V \), there exists a nonzero polynomial \( A'(\xi) \in K[\xi] = K[x_{11}, y_{11}, z_{11}, t_{11}] \) such that

\[
G(\nu) - A'(\xi) \frac{f_0(\nu)}{x_{11}^5} \in (x_{11}^m) \subset K[\xi].
\]
On the other hand, we have

\[ G(n) = \sum_{i+j+k+h \leq m} c'_{ijkh} x_1^{i+j+k} y_1^{j+k} z_1^{i+k} t_1^{k+h}, \]

where \( c'_{ijkh} = c_{ijkl} \) with \( l = m - (i + j + k + h) \). Setting \( B'_4(\zeta) \) the part of \( A'(\zeta) \) with the lowest degree in \( x_{11} \), the assumption \( r_4(G) < m \), the definition of \( G^{(4)}(X) \) given in the previous pages and formula (1) above imply

\[ (2) \quad G^{(4)}(n) = \sum_{i+j+k+h = r_4(G)} c'_{ijkh} x_1^{i+j+k} y_1^{j+k} z_1^{i+k} t_1^{k+h} = B'_4(\zeta) \frac{p_4(n)}{x_1^5}, \]

since \( \sigma_4(\zeta) = i + 2j + k + h \). Going back to \( U_4 \) via \( x_{11} = x \), \( y_{11} = y/x^2 \), \( z_{11} = z/x \), \( t_{11} = t/x \), and then to the original coordinates \( X \), formula (2) becomes

\[ G^{(4)}(X) = \frac{B_4(X)p_4(X)}{X_0^n}, \]

for some \( n \geq 0 \) and a polynomial \( B_4(X) \). Since \( G^{(4)}(X) \) is a homogeneous polynomial and \( X_0 \) does not factorize \( p_4(X) \), we see that \( X_0^n \) has to factorize \( B_4(X) \), thus we may assume that \( n = 0 \) and \( B_4 \) is the homogeneous polynomial of degree \( m - 6 \) we were looking for. The final assertions of the lemma follow from the fact that \( r_4(p_4) = 5 \), \( p_4^{(4)} = p_4 \) and the definition of \( B'_4(\zeta) \).

**Proof of Proposition 2.** By Lemma 1 in Chapter 1, a homogeneous polynomial \( G \) of degree \( m \) defines a global \( m \)-canonical adjoint to \( V \) if and only if \( r_\lambda(G) \geq m \) for each \( 0 \leq \lambda \leq 4 \). Therefore there is \( \lambda \) such that \( r_\lambda(F) < m \).

If \( r_0(F) < m \), Lemma 4 implies that there is a homogeneous polynomial \( B'_0(X) \) of degree \( m - 6 \) such that \( F^{(0)} = B'_0 p_0 \) and \( r_0(F - B'_0 f_0) \geq r_0(F) + 1 \). Repeating the same argument for the value of \( r_0 \) on the polynomial \( F - B'_0 f_0 \), by induction it follows that there exists a homogeneous polynomial \( B_0(X) \) of degree \( m - 6 \) such that \( r_0(F - B_0 f_0) \geq m \), and we set \( F_0 = F - B_0 f_0 \).

If instead \( r_0(F) \geq m \), we set \( F_0 = F \) and \( B_0 = 0 \).

If \( r_1(F_0) < m \), Lemma 4 again implies that there is a homogeneous polynomial \( B'_1 \) of degree \( m - 6 \) such that \( F^{(1)} = B'_1 p_1 \) and \( r_1(F_0 - B'_1 f_0) \geq r_1(F_0) + 1 \). If instead \( r_1(F_0) \geq m \), we set \( F_1 = F_0 \) and \( B_1 = 0 \).
We claim that we still have \( r_0(F_0 - B_1 f_6) \geq m \). Since \( F_0^{(1)} = B_1 p_1 \) is part of \( F_0 \), we have \( r_0(B_1 p_1) \geq m \). Recall that \( r_0(p_1) = 5 \) and note that \( p_0 \) and \( p_1 \) share monomials \( X^5 \) with \( \sigma_0(x) = \sigma_1(x) = 5 \), e.g. \( X_0 X_1^2 X_2 X_3 \). Hence \( r_0(B_1') \geq m - 5 \) and \( r_0(B_1' f_6) \geq m \), which implies our claim.

Repeating the same arguments for the value of \( r_1 \), by induction it follows that there is a homogeneous polynomial \( B_{1} \) of degree \( m - 6 \) such that \( r_1(F_0 - B_1 f_6) \geq m, \lambda = 0, 1 \), and we set \( F_1 = F_0 - B_1 f_6 = F - (B_0 + B_1) f_6 \).

If \( r_2(F_1) < m \), we follow the same steps. Since \( p_2 \) share with both \( p_0, p_1 \) the monomial \( X_0 X_1^2 X_2 X_3 \) (which has \( \sigma_2(x) = 5, \lambda = 0, 1, 2 \)), we see that there is a homogeneous polynomial \( B_2 \) of degree \( m - 6 \) such that \( r_2(F_1 - B_2 f_6) \geq m, \lambda = 0, 1, 2 \), and we set \( F_2 = F_1 - B_2 f_6 = F - (B_0 + B_1 + B_2) f_6 \).

If instead \( r_2(F_1) \geq m \), we set \( F_2 = F_1 \) and \( B_2 = 0 \).

The same arguments apply for \( \lambda = 3 \) and then for \( \lambda = 4 \), noting that \( p_3 \) and \( p_4 \) share with \( p_0 \) the monomial \( X_0^2 X_2^3 X_3^3 X_4 \), with \( p_1 \) the monomial \( X_0^2 X_1^3 X_2^3 X_3^3 \), and with \( p_2 \) the monomial \( X_0^2 X_2^3 X_3^3 X_4 \). Therefore, by following the same steps, we find out homogeneous polynomials \( B_3, B_4 \) such that, setting \( A = B_0 + B_1 + \cdots + B_4 \), we have \( r_4(F - Af_6) \geq m \), \( 0 \leq \lambda \leq 4 \), which is the assertion of Proposition 2 for \( V \).

In case of \( V' \), we just replace \( f_6 \) by \( f_6' \) and \( \sigma_4 \) by the function \( z \mapsto |z| + k - l \), hence \( p_0, \ldots, p_4 \) become

\[
\begin{align*}
p_0 &= a_{30102} X_0^2 X_2 X_3^2 + a_{13020} X_0 X_1^3 X_2 \frac{X_3}{3} + a_{20301} X_0^2 X_2^3 X_4 + a_{20031} X_0^2 X_2 X_4 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{11220} X_0 X_1^2 X_2^2 X_4 \frac{X_3}{3}, \\
p_1 &= a_{13020} X_0 X_2^3 X_3^3 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{02112} X_2^2 X_3^2 X_4, \\
p_2 &= a_{20301} X_0 X_2^3 X_3^3 X_4 + a_{10203} X_0 X_2^3 X_3^3 X_4 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{11220} X_0 X_1^2 X_2^2 X_3 + a_{02112} X_2^2 X_3^2 X_4, \\
p_3 &= a_{13020} X_0 X_2^3 X_3^3 + a_{20031} X_0^2 X_2 X_3^3 X_4 + a_{12210} X_0 X_1^2 X_2^2 X_3 + a_{02112} X_2^2 X_3^2 X_4, \\
p_4 &= a_{30102} X_0 X_2^2 X_4^2 + a_{20031} X_0 X_2^2 X_4^2 + a_{10203} X_0 X_2^2 X_4^2 + a_{02112} X_2^2 X_3^2 X_4,
\end{align*}
\]

and we note that \( p_1, \ldots, p_4 \) share the monomial \( X_0^2 X_2 X_3^3 X_4^2 \); \( p_0, p_1, p_2 \) share the monomial \( X_0 X_2^2 X_3^2 X_4 \) and finally \( p_0, p_3, p_4 \) share the monomial \( X_0^2 X_3^2 X_4 \).

In case of \( V'' \), we replace \( f_6 \) by \( f_6'' \) and \( \sigma_4 \) by the function \( z \mapsto |z| + k - l \), thus \( p_0, \ldots, p_4 \) now become
Examples of Threefolds with Kodaira Dimension 1 or 2

\[ p_0 = a_{30012}X_0^3X_3X_4^2 + a_{13020}X_0X_1^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{20031}X_0^2X_3^3X_4 + a_{22011}X_0^2X_1^3X_3X_4 + a_{12210}X_0X_1^3X_2^3X_3 + a_{11220}X_0X_1X_2^3X_3^2, \]

\[ p_1 = a_{13020}X_0^3X_3^2 + a_{22011}X_0^2X_1^3X_3X_4 + a_{12210}X_0X_1^3X_2^3X_3 + a_{12012}X_0X_1^3X_3X_4 + a_{02112}X_1^3X_2X_3X_4, \]

\[ p_2 = a_{20301}X_0^2X_2^3X_4 + a_{12210}X_0X_1^3X_2^3X_3 + a_{11220}X_0X_1^3X_2^3X_3 + a_{10221}X_0X_1^3X_2X_3^2X_4 + a_{10212}X_0X_1^3X_2X_3^2X_4 + a_{02112}X_1^3X_2X_3X_4, \]

\[ p_3 = a_{13020}X_0^3X_3^2 + a_{20301}X_0^2X_2^3X_4 + a_{10023}X_0X_1^3X_2X_3X_4 + a_{11220}X_0X_1X_2^3X_3^2 + a_{10221}X_0X_1X_2^2X_3^2X_4 + a_{10212}X_0X_1X_2^2X_3^2X_4 + a_{02112}X_1^2X_2X_3X_4, \]

\[ p_4 = a_{30012}X_0^3X_3X_4^2 + a_{20301}X_0^2X_2^3X_4 + a_{12012}X_0X_1^3X_3X_4 + a_{10023}X_0X_1^3X_3X_4^2 + a_{10212}X_0X_1^3X_3^2X_4 + a_{02112}X_1^3X_2X_3X_4^2, \]

and we note that \( p_1, \ldots, p_4 \) share the monomial \( X_0^2X_2X_3X_4^2 \); \( p_0, p_1, p_2 \) share the monomial \( X_0X_1^3X_3^2X_3 \); \( p_0, p_3 \) share \( X_0X_1X_2^3X_3^2 \) and \( p_0, p_4 \) share \( X_0^3X_3X_4^2 \). This concludes the proof of Proposition 2 in all cases.

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