A Threefold with \( p_g = 0 \) and \( P_2 = 2 \)

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**Abstract** - We construct a nonsingular threefold \( X \) with \( q_1 = q_2 = p_g = 0 \) and \( P_2 = 2 \) whose \( m \)-canonical transformation \( \varphi_{|mK_X|} \) has the following properties

i) \( \varphi_{|mK_X|} \) has the generic fiber of dimension \( \geq 1 \), for \( 2 \leq m \leq 5 \);

ii) it is generically a transformation \( 2 : 1 \), for \( 6 \leq m \leq 8 \) and \( m = 10 \);

iii) it is birational for \( m = 9 \) and \( m \geq 11 \).

So, we have a gap for \( m = 10 \) in the birationality of \( \varphi_{|mK_X|} \).

**Introduction.**

In the classification of nonsingular varieties \( X \) of general type, the \( m \)-canonical transformation \( \varphi_{|mK_X|} \), where \( K_X \) is a canonical divisor on \( X \), plays an important part. The main problem concerning \( \varphi_{|mK_X|} \) regards its birationality. The property of \( \varphi_{|mK_X|} \) to have the generic fiber given by a finite set of points is important too.

In the case where \( X \) is a threefold, Meng Chen has given several limitations for the birationality of \( \varphi_{|mK_X|} \). In the particular case where \( X \) has the geometric genus \( p_g \geq 2 \), Chen ([Che2], [Che3]) proved that:

- if \( p_g \geq 4 \), then \( \varphi_{|mK_X|} \) is birational for \( m \geq 5 \);
- if \( p_g = 3 \), then \( \varphi_{|mK_X|} \) is birational for \( m \geq 6 \);
- if \( p_g = 2 \), then \( \varphi_{|mK_X|} \) is birational for \( m \geq 8 \).

Such limitations are optimal, as demonstrated by examples costructed by Chen himself [Che2] if \( p_g \geq 4 \), by S. Chiaruttini - R. Gattazzo ([CG]) if \( p_g = 3 \), by S. Chiaruttini ([Chi]) and by C. Hacon, considering an example of M. Reid [Re], if \( p_g = 2 \) (see [Che3]).

In the case of \( p_g = 1 \) and \( p_g = 0 \), we have only partial results and the

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problem of finding an optimal limitation for the birationality of \( \varphi_{|mK_X|} \) remains ([Che1]). If \( p_g = 1 \) and the bigenus of \( X \) is \( P_2 = 2 \), then a Chen-Zuo’s limitation ([CZ]) states that \( \varphi_{|mK_X|} \) is birational for \( m \geq 11 \). We constructed ([S1]) a threefold \( X \) with \( q_1 = q_2 = 0 \) (where \( q_1 \) and \( q_2 \) are the first and second irregularities of \( X \)) \( p_g = 1 \) and \( P_2 = 2 \) such that \( \varphi_{|mK_X|} \) is birational if and only if \( m \geq 11 \), (cf. also \( X_{22} \) in [Re], p. 359, and [F]); so the above limitation is optimal.

As for threefolds with \( p_g = 0 \), we tried to find examples of \( X \) with \( q_1 = q_2 = 0, P_2 = 2 \) and with the birationality of \( \varphi_{|mK_X|} \) for \( m \) large. The results obtained were worse than expected as regards the birationality of \( \varphi_{|mK_X|} \), while an interesting result emerged for the gaps in the birationality of \( \varphi_{|mK_X|} \). Having obtained the birationality of \( \varphi_{|mK_X|} \) if and only if \( m \geq 11 \) in the case of \( p_g = 1 \) and \( P_2 = 2 \), the expected result in the new case of \( p_g = 0 \) and \( P_2 = 2 \) is birationality if and only if \( m > 11 \). Instead, all our constructions of threefolds \( X \) with \( q_1 = q_2 = p_g = 0 \) and \( P_2 = 2 \) have the 9-canonical transformation \( \varphi_{|9K_X|} \), which is birational, but some of them also have \( \varphi_{|10K_X|} \), which is not birational, and \( \varphi_{|mK_X|} \), which is birational if and only if \( m = 9 \) and \( m \geq 11 \).

So, the threefolds with this property have a gap in the birationality of \( \varphi_{|mK_X|} \) for \( m = 10 \). This came as a surprise because the only cases of gaps in the birationality of \( \varphi_{|mK_X|} \) that we found were in threefolds with \( q_1 = q_2 = p_g = P_2 = P_3 = 0 \) or \( q_1 = q_2 = p_g = P_2 = 0 \). Such examples with gaps are in [S3], where an example is constructed with the same properties as the example \( X_{46} \) in Reid’s list ([Re]), and in [Ro2].

In the present paper, we construct a threefold \( X \) with the properties described – i.e. \( \varphi_{|mK_X|} \) is birational if and only if \( m = 9 \) and \( m \geq 11 \), \( q_1 = q_2 = 0 \) and \( p_g = 0, P_2 = 2 \) – and with further plurigenera \( P_3 = 2, P_4 = P_5 = 4, P_6 = P_7 = 8, P_8 = 13, P_9 = 15, P_{10} = 19, P_{11} = 22 \).

We note that \( X \) is birationally distinct from the threefolds appearing in the lists of [Re], pp. 358-359 and [F], pp. 151-154, 169-170, because \( X \) has different plurigenera from those of the threefolds in said lists.

The example \( X \) is constructed as a desingularization of a degree six hypersurface \( V \subset \mathbb{P}^4 \) endowed with a singularity at each of the five vertices \( A_0, A_1, A_2, A_3 \) and \( A_4 \) of the fundamental pentahedron. The construction is similar to those in [S1]. Precisely, we put a triple point with an infinitely-near double surface at \( A_0 \) on \( V \), we put a triple point with an infinitely-near triple curve at \( A_1, A_2, A_3 \), and an ordinary 4-ple point at \( A_4 \). Other unimposed singularities appear on \( V \), but they do not affect the birational invariants of \( X \).

The ground field \( k \) is an algebraically closed field of characteristic zero, which we can assume to be the field of complex numbers.
1. Imposing singularities on a degree six hypersurface $V$ in $\mathbb{P}^4$.

Let $(x_0, x_1, x_2, x_3, x_4)$ be homogeneous coordinates in $\mathbb{P}^4$ and let us indicate as $f_6(X_0, X_1, X_2, X_3, X_4)$ a form (homogeneous polynomial) of degree 6, in the variables $X_0, X_1, X_2, X_3, X_4$, defining a hypersurface $V \subset \mathbb{P}^4$ of degree six. We impose a triple point on $V$ at each of the four vertices $A_0 = (1, 0, 0, 0, 0)$, $A_1 = (0, 1, 0, 0, 0)$, $A_2 = (0, 1, 0, 0, 0)$, $A_3 = (0, 0, 1, 0, 0)$ and an ordinary 4-ple (quadruple) point at $A_4 = (0, 0, 0, 0, 1)$ of the fundamental pentahedron $X_0 X_1 X_2 X_3 X_4 = 0$.

The equation for $V$, with the imposed singularities, is of the following type

$$V : f_6(X_0, X_1, X_2, X_3, X_4) = X_0^3(a_{33000}X_1^3 + \cdots) + X_0^3(a_{23100}X_2^3X_2 + \cdots) + X_3^3(\cdots) + X_0^3(\cdots) + a_{22200}X_0^2X_1^2X_2^2 + a_{22110}X_0^2X_1^2X_2X_3 + \cdots + a_{00022}X_0^2X_3^2X_4^2 = 0,$$

where $a_{ijklm} \in k$ denotes the coefficient of the monomial $X_0^iX_1^jX_2^kX_3^lX_4^m$.

Moreover, we impose a double surface $S_0$ infinitely near $A_0$ in the first neighbourhood. We impose the same double surface $S_0$, which is locally isomorphic to a plane as in $[S_2]$. In addition, we impose a triple curve $C_i$ infinitely near $A_i$, $i = 1, 2, 3$ in the first neighbourhood. $C_i$ is locally isomorphic to a straight line as in $[S_1]$.

As an example, we provide a few details on the realization of the singularity at $A_0$ on $V$. This will also enable a better understanding in the sequel of the computation of the $m$-canonical adjoints to $V$ and of the $m$-genus $P_m$ of a desingularization $X$ of $V$, $\sigma : X \rightarrow V$ (cf. section 5). Let us consider the affine open set $U_0 \ni A_0$ in $\mathbb{P}^4$ given by $X_0 \neq 0$ of affine coordinates

$$\left( x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}, z = \frac{X_3}{X_0}, t = \frac{X_4}{X_0} \right).$$

The affine equation of $V \cap U_0$ is given by $f_6(1, x, y, z, t) = 0$.

The affine coordinates of $A_0$ are $(0, 0, 0, 0)$, so the blow-up of $\mathbb{P}^4$ at the point $A_0$ is locally given by the formulas:

$$B_{x_1} : \begin{cases} x = x_1 & y = x_1 y_1 \\ z = x_1 z_1 & t = x_1 t_1 \end{cases}$$

$$B_{y_1} : \begin{cases} x = x_2 y_2 & y = y_2 \\ z = y_2 z_2 & t = y_2 t_2 \end{cases}$$

$$B_{z_1} : \begin{cases} x = x_3 z_3 & y = y_3 z_3 \\ z = z_3 & t = z_3 t_3 \end{cases}$$

$$B_{t_1} : \begin{cases} x = x_4 t_4 & y = y_4 t_4 \\ z = z_4 t_4 & t = t_4 \end{cases}$$

and we consider $B_{t_4}$. The strict (or proper) transform $V'$ of $V$ with respect to
the local blow-up $B_{t_4}$ has an affine equation given by

$$V': \frac{1}{t_4^3} f_0(1, x_4 t_4, y_4 t_4, z_4 t_4, t_4) = a_{31200} x_4 y_4^2 + \cdots + a_{00222} y_4^2 z_4^2 t_4^3 = 0.$$ 

On this threefold $V'$ we impose the plane $S_0 \cap U_0$ given affinely by

$$\begin{cases} 
  x_4 = 0 \\
  t_4 = 0
\end{cases}$$

as a singular plane of multiplicity two (i.e. as a double plane). The conditions on the coefficients $a_{ijkl}$, such that $V$ has the double plane $S_0 \cap U_0$ infinitely near $A_0$, are given by

$$
\begin{align*}
  a_{31200} &= 0 & a_{30210} &= 0 & a_{30012} &= 0 & a_{20220} &= 0 \\
  a_{31110} &= 0 & a_{30201} &= 0 & a_{30003} &= 0 & a_{20211} &= 0 \\
  a_{31101} &= 0 & a_{30120} &= 0 & a_{20310} &= 0 & a_{20202} &= 0 \\
  a_{31020} &= 0 & a_{30111} &= 0 & a_{20301} &= 0 & a_{20121} &= 0 \\
  a_{31011} &= 0 & a_{30102} &= 0 & a_{20130} &= 0 & a_{20112} &= 0 \\
  a_{31002} &= 0 & a_{30003} &= 0 & a_{20003} &= 0 & a_{20002} &= 0. \\
  a_{30300} &= 0 & a_{30021} &= 0
\end{align*}
$$

In much the same way as above and precisely as in $[S_1]$, we impose a triple curve $C_i$ infinitely near $A_i$ and in the first neighbourhood, which is locally isomorphic to a straight line, for $i = 1, 2, 3$. Further information on the above singularities can be found in $[S_4]$.

We give the final equation for our hypersurface $V$ after imposing all the above-mentioned singularities. We have chosen several coefficients as equal to zero because they are inessential for the computation of the birational invariants of a desingularization $\sigma : X \to V$ of $V$. The shortest equation with the essential coefficients is

$$V : f_0(X_0, X_1, X_2, X_3, X_4)$$

$$= a_{33000} X_0^3 X_1^3 + a_{32100} X_0^3 X_1^2 X_2 + a_{32001} X_0^3 X_1^2 X_3 + a_{23010} X_0^2 X_1^3 X_3 + a_{13020} X_0 X_1^2 X_3^2 + a_{10302} X_0 X_1^2 X_2^2 + a_{03030} X_1^3 X_3 + a_{02031} X_1^2 X_3^2 X_4 + a_{01032} X_1 X_3^3 X_4 + a_{22000} X_0^2 X_1^2 X_2^2 + a_{22020} X_0^2 X_1^2 X_3^2 + a_{22002} X_0 X_1^2 X_2^2 X_4 + a_{21210} X_0^2 X_1 X_2^2 X_3 + a_{21201} X_0^2 X_1^2 X_2 X_4 + a_{21102} X_0^2 X_1 X_2 X_3^2 + a_{21021} X_0^2 X_1 X_3 X_4 + a_{21012} X_0^2 X_2 X_3 X_4 + a_{12012} X_0 X_1^2 X_3 X_4 + a_{02022} X_1^2 X_3^2 X_4 + a_{00222} X_0^2 X_3^2 X_4 = 0.$$ 

From here on, $V$ denotes this last hypersurface defined by the above form $f_0(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters $a_{ijkl}$. As a reminder of this generic choice, we sometimes call $V$: the generic $V$. 

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2. **Imposed and unimposed singularities of \( V \): the actual singularities.**

We consider the hypersurface \( V \) given at the end of section 1.

New unimposed singularities appear on the (generic) \( V \) close to the singularities imposed on \( V \); they are actual or infinitely-near singularities. We call a singularity on \( V \) actual to distinguish it from those which are infinitely near. We call a singularity of \( V \) unimposed if it does not appear in the list of singularities in section 1.

There are six unimposed actual double (straight) lines on \( V \) given by \( A_0A_2, A_0A_3, A_0A_4, A_1A_2, A_1A_4, A_2A_3 \) and the unimposed double plane cubic

\[
\begin{aligned}
    X_1 &= 0 \\
    X_2 &= 0 \\
    a_{01032} X_3^2 X_4 + a_{21021} X_0^2 X_3 + a_{21012} X_0^2 X_4 &= 0.
\end{aligned}
\]

The generic \( V \) has no other actual singularities. It follows that the generic \( V \) is reduced, irreducible and normal.

The cubic lies on the plane \( \{ X_1 = 0 \} \) \( \cup \{ X_2 = 0 \} \), which is simple on \( V \). The picture of the six double lines is as follows, where the double lines are drawn in bold type.

![Diagram of singularities](image)

3. **The infinitely-near singularities of \( V \).**

In section 2, we described the actual singularities on \( V \); in the present section, we briefly describe the infinitely-near singularities. Here again, new infinitely-near singularities appear on the generic \( V \) alongside the infinitely-near singularities imposed on \( V \). They are only double singular curves and isolated double points, so none of the unimposed singularities (be they actual or otherwise) affect the birational invariants of a desingularization \( \sigma : X \rightarrow V \) of \( V \), such as the irregularities and the plurigenera.
of $X$. This means that, in calculating these invariants, we can assume that there are only the imposed singularities on $V$.

We compute said birational invariants of $X$ using the theory of adjoints and pluricanonical adjoints developed in [S$_1$]. We can apply this theory because the singularities on the hypersurface $V$ satisfy the hypotheses of [S$_1$], i.e. it must be possible to resolve the singularities on $V$ with local blow-ups along linear affine subspaces; moreover, the degree six hypersurfaces in $\mathbb{P}^4$ must have singularities of codimension $\geq 2$ (i.e. the hypersurfaces must be normal).

Such hypotheses on the singularities are satisfied by either actual or infinitely-near singularities of $V$. In particular, $V$ is normal (section 2). To be precise, all the singularities of $V$ are resolved with local blow-ups either along straight lines, that are double on $V$ and on strict transforms of $V$, or along planes containing double curves and points. These planes are simple on $V$ and on strict transforms of $V$, e.g. the simple plane \[
\begin{aligned}
X_1 &= 0 \\
X_2 &= 0
\end{aligned}
\] containing the cubic curve on $V$ in section 2.

Having said as much, we only give details on the imposed infinitely-near singularities of $V$ that are needed in the sequel.

From section 1, we already have the information that we need about the triple point $A_0$ and the double surface $S_0$ infinitely near $A_0$.

Next, we consider the triple point $A_1$ on $V$ and the blow-up at $A_1$. Let us consider the affine open set $U_1 \ni A_1$ in $\mathbb{P}^4$ given by $X_1 \neq 0$ of affine coordinates \[
\begin{aligned}
x &= \frac{X_0}{X_1}, \\
y &= \frac{X_2}{X_1}, \\
z &= \frac{X_3}{X_1}, \\
t &= \frac{X_4}{X_1}
\end{aligned}
\] The affine equations of $V \cap U_1$ are given by $f_6(x, 1, y, z, t) = 0$. The affine coordinates of $A_1$ are $(0, 0, 0, 0, 0)$.

We can assume that the blow-up at $A_1$ is the first to be performed, so we can use the local blow-ups $B_{x_1}, B_{y_2}, B_{z_3}, B_{t_4}$ in section 1.

The strict transform of $V \cap U_1$, with respect to $B_{t_4}$, is given by

- \[ V'_{t_4} : \frac{1}{t_4^3} f_6(x_4 t_4, 1, y_4 t_4, z_4 t_4, t_4) \]

\[ = a_{33000} x_4^3 + \cdots + a_{03030} z_4^3 + \cdots + a_{12012} x_4 z_4 t_4 + \cdots = 0. \]

We are interested in the triple curve infinitely near $A_1$. So, we focus locally on the triple line on $V'_{t_4}$ belonging to the exceptional divisor $t_4 = 0$ of the local blow-up $B_{t_4}$. This triple line is given by \[
\begin{aligned}
x_4 &= 0 \\
z_4 &= 0 \\
t_4 &= 0
\end{aligned}
\]

Let us go on to consider the triple point $A_2$ on $V$, the blow-up at $A_2$ and the affine open set $U_2 \ni A_2$ in $\mathbb{P}^4$ given by $X_2 \neq 0$ of affine coordinates
\( (x = X_0 X_2, \ y = X_1 X_2, \ z = X_3 X_2, \ t = X_4 X_2) \). The affine equations of \( V \cap U_2 \) are given by \( f_6(x, y, 1, z, t) = 0 \). The affine coordinates of \( A_2 \) are \((0, 0, 0, 0)\).

Here again, we can assume that the blow-up at \( A_2 \) is the first to be performed, so we can use the local blow-ups \( B_{x_2}, B_{y_2}, B_{z_2}, B_{t_2} \) in section 1.

The strict transform of \( V \cap U_2 \), with respect to \( B_{y_2} \), is given by

\[
V'_{y_2} : \frac{1}{y_2^3} f_6(x_2 y_2, y_2, 1, y_2 z_2, y_2 t_2) \\
= a_{10302} x_2 ^2 y_2 + \cdots + a_{22200} x_2 ^2 y_2 + \cdots + a_{002022} y_2 z_2 ^2 t_2 ^2 = 0.
\]

We are interested in the triple curve infinitely near \( A_2 \), so we focus locally on the triple line on \( V'_{y_2} \) belonging to the exceptional divisor \( y_2 = 0 \) of the local blow-up \( B_{y_2} \). This triple line is given by

\[
\begin{align*}
x_2 &= 0 \\
y_2 &= 0 \\
t_2 &= 0
\end{align*}
\]

Finally, let us consider the triple point \( A_3 \) on \( V \), the blow-up at \( A_3 \) and the affine open set \( U_3 \ni A_3 \) in \( \mathbb{P}^4 \) given by \( X_3 \neq 0 \) of affine coordinates

\[
(x = X_0 X_3, \ y = X_1 X_3, \ z = X_2 X_3, \ t = X_4 X_3) \). The affine equations of \( V \cap U_3 \) are given by \( f_6(x, y, z, 1, t) = 0 \). The affine coordinates of \( A_3 \) are \((0, 0, 0, 0)\).

We can again assume that the blow-up at \( A_3 \) is the first to be performed, so we can use the local blow-ups \( B_{x_1}, B_{y_2}, B_{z_2}, B_{t_2} \) in section 1.

The strict transform of \( V \cap U_3 \), with respect to \( B_{x_1} \), is given by

\[
V'_{x_1} : \frac{1}{x_1 ^3} f_6(x_1 y_1, x_1 z_1, 1, x_1 t_1) \\
= a_{03030} y_1 ^3 + \cdots + a_{22020} x_1 y_1 ^2 + \cdots + a_{21021} x_1 y_1 t_1 + \cdots = 0.
\]

We are interested in the triple curve infinitely near \( A_3 \), so we focus locally on the triple line on \( V'_{x_1} \) belonging to the exceptional divisor \( x_1 = 0 \) of the local blow-up \( B_{x_1} \). This triple line is given by

\[
\begin{align*}
x_1 &= 0 \\
y_1 &= 0 \\
t_1 &= 0
\end{align*}
\]

To end this section, we add one more item of information, drawing the picture of the tree of local blow-ups resolving the singularity at \( A_0 \) and those infinitely near.
where “ns” means “nonsingular”.

4. The m-canonical adjoints to \( V \subset \mathbb{P}^4 \).

Let

\[
P_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 = \mathbb{P}^4
\]

be a sequence of blow-ups solving the singularities of \( V \).

If we call \( V_i \subset P_i \) the strict transform of \( V_{i-1} \) with respect to \( \pi_i \), then the above sequence gives us

\[
X = V_r \xrightarrow{\pi'_r} \cdots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,
\]

where \( \pi'_i = \pi_i|_{V_i} : V_i \to V_{i-1} \) and \( \sigma_{\chi} : X \to V \), \( \sigma = \pi_r \circ \cdots \circ \pi_1 \), is a desingularization of \( V \subset \mathbb{P}^4 \).
Let us assume that $\pi_i$ is a blow-up along a subvariety $Y_{i-1}$ of $\mathbb{P}_{i-1}$, of dimension $j_{i-1}$, which can be either a singular or a nonsingular subvariety of $V_{i-1} \subset \mathbb{P}_{i-1}$ (i.e. $Y_{i-1}$ is a locus of singular or simple points of $V_{i-1}$). Let $m_{i-1}$ be the multiplicity of the variety $Y_{i-1}$ on $V_{i-1}$.

Let us set $n_{i-1} = -3 + j_{i-1} + m_{i-1}$, for $i = 1, \ldots, r$ and $\deg(V) = d$.

A hypersurface $\Phi_{m(d-5)}$ of degree $m(d-5)$, $m \geq 1$, in $\mathbb{P}^4$ is an $m$-canonical adjoint to $V$ (with respect to the sequence of blow-ups $\pi_1, \ldots, \pi_r$) if the restriction to $X$ of the divisor

$$D_m = \pi_r^* \{ \pi_{r-1}^* \cdots \pi_1^*(\Phi_{m(d-5)}) - mn_0 E_1 \cdots - mn_{r-2} E_{r-1} \} - mn_{r-1} E_r,$$

is effective, i.e. $D_m|_X \geq 0$, where $E_i = \pi^{-1}(Y_{i-1})$ is the exceptional divisor of $\pi_i$ and $\pi_i^* : \text{Div}(\mathbb{P}_{i-1}) \rightarrow \text{Div}(\mathbb{P}_i)$ is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S1], sections 1.2).

An $m$-canonical adjoint $\Phi_{m(d-5)}$ is an global $m$-canonical adjoint to $V$ (with respect to $\pi_1, \ldots, \pi_r$) if the divisor $D_m$ is effective on $\mathbb{P}_r$, i.e. $D_m \geq 0$ (loc. cit.).

Note that, if $\Phi_{m(d-5)}$ is an $m$-canonical adjoint to $V$, then $D_m|_X \equiv mK$, where ‘$\equiv$’ denotes linear equivalence and $K$ denotes a canonical divisor on $X$.

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that $\pi_1$ is the blow-up at the triple point $A_0$, $\pi_2$ is the blow-up along the double surface $S_0$ infinitely near $A_0$, $\pi_3$ is the blow-up at the triple point $A_1$, $\pi_4$ is the blow-up along the triple curve $C_1$ infinitely near $A_1$, $\pi_5$ is the blow-up at the triple point $A_2$, $\pi_6$ is the blow-up along the triple curve $C_2$ infinitely near $A_2$, $\pi_7$ is the blow-up at the triple point $A_3$, $\pi_8$ is the blow-up along the triple curve $C_3$ infinitely near $A_3$ and the blow-up $\pi_9$ is the one at the 4-ple point $A_4$.

The example $V$ has degree $d = 6$ and $D_m$, relative to our $X$, is given by:

$$(*) \quad D_m = \pi_r^* \cdots \pi_3^* \{ \pi_2^*(\Phi_{m}) - mE_2 \} - mE_4 - mE_6 - mE_8 - mE_9,$$

where $E_i$ is the exceptional divisor of the blow-up $\pi_i$ and, to be more specific, $E_1$ is the exceptional divisor of the blow-up $\pi_1$ at the triple point $A_0$, $E_2$ is the exceptional divisor of the blow-up $\pi_2$ along $C_1$, ... and $E_9$ is the exceptional divisor of the blow-up $\pi_9$ at the 4-ple point $A_4$.

No other exceptional divisors are subtracted in $D_m$ because, as we said before, the unimposed singularities are either actual or infinitely-near double singular curves or isolated double points on our (generic) $V$. Put more precisely, the exceptional divisors of the blow-ups along the double curves appear with coefficient $n_i = 0$ in the above expression of $D_m$ and the exceptional divisors of the blow-ups along simple planes appear again with
coefficient \( n_j = 0 \). Since we have resolved all the unimposed singularities with blow-ups either along double curves or along simple planes, only the exceptional divisors \( E_2, E_4, E_6, E_8 \) and \( E_9 \) appear in \( D_m \). Note, moreover, that the exceptional divisor of a blow-up at a triple point also appears with coefficient \( n_h = 0 \) in \( D_m \).

5. The plurigenera of a desingularization \( X \) of \( V \).

Let us consider the equation of \( V \):\( f_6(X_0, X_1, X_2, X_3, X_4) = 0 \) at the end of section 1 and arrange the form \( f_6 \) according to the powers of \( X_4 \).

\((**)\) \[ f_6 = \phi_4(X_0, X_1, X_2, X_3)X_4^2 + \psi_5(X_0, X_1, X_2, X_3)X_4 + \phi_6(X_0, X_1, X_2, X_3), \]

where \( \phi_i(X_0, X_1, X_2, X_3) \) is a form of degree \( i \) in \( X_0, X_1, X_2, X_3 \) and precisely

\[ \phi_4(X_0, X_1, X_2, X_3) = a_{1002}X_0X_2^3 + a_{0100}X_1X_3^3 + a_{2200}X_0^2X_1^2 + \cdots + a_{0022}X_2X_3^2. \]

Next, let us consider the hypersurface \( \Phi_m \), appearing in \((*)\) section 4 and assume that its equation is \( F_m(X_0, X_1, X_2, X_3, X_4) = 0 \), of degree \( m \). Arranging the form \( F_m \) according to the powers of \( X_4 \), we can write

\((***)\) \[ F_m(X_0, X_1, X_2, X_3, X_4) = \psi_s(X_0, X_1, X_2, X_3)X_4^{m-s} + \psi_{s+1}(X_0, X_1, X_2, X_3)X_4^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_2, X_3), \]

where \( \psi_j(X_0, X_1, X_2, X_3) \) is a form of degree \( j \) in \( X_0, X_1, X_2, X_3 \) and \( s \) is an integer satisfying \( 0 \leq s \leq m \).

Under the sole hypothesis that \( V \) has a 4-ple point at \( A_4 \) the following lemma holds.

**Lemma 1.** With the above notations, if \( \Phi_m \) is an \( m \)-canonical adjoint (be it global or not), then, modulo \( V : f_6 = 0 \), we can assume that \( s \geq m - 1 \) in \((***)\); i.e. if \( \Phi_m : F_m = 0 \) is an \( m \)-canonical adjoint, then we can assume that

\[ F_m = \psi_{m-1}(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3). \]

Moreover, we have the equality

\[ \psi_{m-1}(X_0, X_1, X_2, X_3) = A_{m-5}(X_0, X_1, X_2, X_3)\phi_4(X_0, X_1, X_2, X_3), \]

where \( A_{m-5}(X_0, X_1, X_2, X_3) \) is a form of degree \( m - 5 \) in \( X_0, X_1, X_2, X_3 \) and \( \phi_4(X_0, X_1, X_2, X_3) \) is defined above in \((**)\).
A threefold with $p_g = 0$ and $P_2 = 2$

The idea for the proof of the above lemma came from M. C. Ronconi [CR], [Ro1]. A detailed proof can be found in [S4] (Lemma 1, section 5).

**Remark 1.** In Lemma 1, we have $F_m = A_{m-5} \psi_4 X_4 + \psi_m$. We see that, if $A_{m-5} = 0$, then $F_m$ defines a global $m$-canonical adjoint $\Phi_m$ to $V$, whereas if $A_{m-5} \neq 0$, then $\Phi_m$ is a “non-global” $m$-canonical adjoint to $V$. The non-global $m$-canonical adjoints to $V$ are important for establishing the birationality of the $m$-canonical transformation $\varphi_{[mK_X]}$ (see next section).

The following lemma is proved in [S4], Lemma 2, section 12, where the singularities at three fundamental points on a degree six hypersurface $V'$ differ from those on $V$ in the present case. More precisely, $V'$ has three triple points with an infinitely-near double plane, whereas $V$ has three triple points with an infinitely-near triple curve. But the proof remains the same in both cases.

**Lemma 2.** The $m$-canonical adjoint to $V$ given by

$$
\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3)\psi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3, X_4) = 0,
$$

has the following property

$$
D_{m|_X} \geq 0 \iff D_m + E_9 \geq 0,
$$

where $D_m = \pi_1^* \cdots \pi_3^*[\pi_2^* \pi_1^*(\Phi_m)] - mE_2 - mE_4 - mE_5 - mE_8 - mE_9$, is defined in (*) section 4.

**Remark 2.** Roughly speaking, the result in Lemmas 1 and 2, that permits us an easy computation of the $m$-genus $P_m$ (or $m$) of a desingularization $\sigma : X \to V$ of $V$, is the following. Our degree six hypersurface $V$ has a 4-ple point, so from Lemma 1 we can assume that the $m$-canonical adjoint $\Phi_m$ is defined by a form of the type $F_m = A_{m-5} \psi_4 X_4 + \psi_m$, where the variable $X_4$ appears to the power 1. In order to compute the linear conditions given by the other singularities to the hypersurfaces $\Phi_m$ so that they are $m$-canonical adjoints to $V$, i.e. to obtain $D_{m|_X} \geq 0$, we find that we do not need to restrict $D_m$ to $X$ and, after imposing $D_{m|_X} \geq 0$, we only need to have $D_m + E_9 \geq 0$. This follows from the fact that $F_m$ contains the variable $X_4$ to the power 2, whereas the form $f_6$ defining $V$ contains the variable $X_4$ to the power 2, and also from the particular singularities obtained in our examples. We note that $E_9$ has to be added to $D_m$, otherwise $D_m$ may not be effective (when $A_{m-5} \neq 0$, ...
see Remark 1). So it is very easy to compute the conditions on $F_m$ such that $D_m + E_9$ is effective and, since $P_m = \text{number of linearly independent forms contained in } F_m$ (cf. [S1]), the computation of $P_m$, $\forall m$, is very easy too.

Now, we are ready to compute the plurigenera of a desingularization $\sigma : X \rightarrow V$ of $V$. Let us write

$$A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \left( \sum_{i+j+k+m-5} a_{ijkh} X_0^i X_1^j X_2^k X_3^h \right) X_4,$$

$$\psi_m(X_0, X_1, X_2, X_3) = \sum_{i' + j' + k' + l' = m} b_{ijkh} X_0^{i'} X_1^{j'} X_2^{k'} X_3^{h'},$$

where $a_{ijkh}, b_{ijkh} \in k$.

- **First let us consider the two blows-up $\pi_1$ and $\pi_2$.** We know that the blow-up $\pi_1$ of $\mathbb{P}^4$ at $A_0$ is given by $\mathcal{B}_{x_1}, \mathcal{B}_{x_2}, \mathcal{B}_{z_1}, \mathcal{B}_{t_1}$ (cf. section 1). Let us consider the affine open set $U_0 = \{X_0 \neq 0\}$ as in section 1.

The total transform of $\Phi_m \cap U_0$ with respect to $\mathcal{B}_{t_4}$ is given by

$$\mathcal{B}_{t_4}^+ (\Phi_m \cap U_0) : A_{m-5}(1, x_4 t_4, y_4 t_4, z_4 t_4) \psi_4(1, x_4 t_4, y_4 t_4, z_4 t_4) t_4 +$$

$$\psi_m(1, x_4 t_4, y_4 t_4, z_4 t_4, t_4) = 0.$$

The double surface $S_0$ infinitely near $A_0$ in affine coordinates $(x_4, y_4, z_4, t_4)$ is given by $\begin{cases} x_4 = 0 \\ t_4 = 0 \end{cases}$ (cf. section 1).

The blow-up $\pi_2$ along $S_0$ is locally given by the formulas:

$$\mathcal{B}_{x_41} : \begin{cases} x_4 = x_{41} \\ y_4 = y_{41} \\ z_4 = z_{41} \\ t_4 = x_{41} t_{41} \end{cases} ; \quad \mathcal{B}_{t_42} : \begin{cases} x_4 = x_{42} t_{42} \\ y_4 = y_{42} \\ z_4 = z_{42} \\ t_4 = t_{42} \end{cases}$$

The total transform of $\mathcal{B}_{t_4}^+ (\Phi_m \cap U_0)$ with respect to $\mathcal{B}_{x_{41}}$ is given by

$$\mathcal{B}_{x_{41}}^+ [\mathcal{B}_{t_4}^+ (\Phi_m \cap U_0)] :$$

$$A_{m-5}(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) \psi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41} +$$

$$\psi_m(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}, x_{41} t_{41}) = 0.$$
A threefold with $p_g = 0$ and $P_2 = 2$

With the above notations, this total transform is given by

$$B^*_{x_{41}} [B^*_{t_4} (\Phi_m \cap U_0)] :$$

$$
\left( \sum_{i+j+k+h=m-5} a_{ijkh} x_{41}^{2j+k+h} y_{41}^k z_{41}^h \right) \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41}
+ \sum_{i'+j'+k'+h'=m} b_{ijkh} x_{41}^{2j'+k'+h'} y_{41}^{k'} z_{41}^{h'} = 0.
$$

The following claims hold true; they are corollaries to Lemma 1 and 2 and consequences of the desingularization of $V$.

**Claim 1.** The composition of the two local blows-up $B_{x_{41}} \circ B_{t_4}$ coincides, up to isomorphisms, with the desingularization $\sigma_x$ on the affine open set $V_{x_{41}}$, because $V_{x_{41}}$ is nonsingular (see the tree of blow-ups at the end of section 3). In fact, $V_{x_{41}}$ is isomorphic to an open set on $X$ and the two above morphisms can be identified on $V_{x_{41}}$.

**Claim 2.** Since $\Phi_m$ is an $m$-canonical adjoint to $V$, by definition we have $D_{m|X} \geq 0$; so, from Lemma 2, we can say that: $D_m + E_9 \geq 0$.

**Claim 3.** From Claims 1 and 2, we deduce (up to isomorphisms) that

$$B^*_{x_{41}} [B^*_{t_4} (\Phi_m \cap U_0)] - mE_2 + E_9 \geq 0.$$

This last inequality is equivalent to the following equality of polynomials

$$
\left( \sum_{i+j+k+h=m-5} a_{ijkh} x_{41}^{2j+k+h} y_{41}^k z_{41}^h \right) \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41}
+ \sum_{i'+j'+k'+h'=m} b_{ijkh} x_{41}^{2j'+k'+h'} y_{41}^{k'} z_{41}^{h'} = x_{41}^m(...)
$$

**Claim 4.** Since $\varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) = x_{41}^3(...)$, the latter equality of polynomials is equivalent to the inequalities

$$
\begin{cases}
2j + k + h + 3 + 1 \geq m & \text{i.e. } j \geq i + 1 \\
2j' + k' + h' \geq m & j' \geq i'
\end{cases}
$$

---

**Next, let us consider the two blow-ups $\pi_3$ and $\pi_4$.** As in section 3, we can assume that the first blow-up that we perform is $\pi_3$ at $A_1$, so we can use the local blows-up $B_{x_1}, B_{y_2}, B_{z_3}, B_{t_4}$ in section 1.

As in the above case of $\pi_1$ and $\pi_2$, here too for $\pi_3$ and $\pi_4$, we find that the
total transform of $\Phi_m \cap U_1$ with respect to $B_{t_4}$ is given by

$$B_{t_4}^*(\Phi_m \cap U_1) : A_{m-5}(x_4t_4, 1, y_4t_4, z_4t_4)\varphi_4(x_4t_4, 1, y_4t_4, z_4t_4) + \psi_m(x_4t_4, 1, y_4t_4, z_4t_4, t_4) = 0.$$  

The triple curve $C_1$ infinitely near $A_1$ in affine coordinates $(x_4, y_4, z_4, t_4)$ is given by (section 3)

$$\begin{cases} 
  x_4 = 0 \\
  z_4 = 0 \\
  t_4 = 0 
\end{cases}$$

The blow-up $\pi_4$ along $C_1$ is locally given by the formulas:

$$\begin{align*}
  B_{x_4}^1 : & x_4 = x_4^{11} \\
  & y_4 = y_4^{11} \\
  & z_4 = x_4^{11}z_4^{11} \\
  & t_4 = x_4^{11}t_4^{11}
\end{align*}$$

$$\begin{align*}
  B_{z_4}^1 : & x_4 = x_4^{22}z_4^{22} \\
  & y_4 = y_4^{22}z_4^{22} \\
  & z_4 = z_4^{22} \\
  & t_4 = z_4^{22}t_4^{22}
\end{align*}$$

$$\begin{align*}
  B_{t_4}^1 : & x_4 = x_4^{33}t_4^{33} \\
  & y_4 = y_4^{33}z_4^{33} \\
  & z_4 = z_4^{33} \\
  & t_4 = t_4^{33}
\end{align*}$$

The total transform of $B_{t_4}^*(\Phi_m \cap U_1)$ with respect to $B_{x_4}^1$ is given by

$$B_{x_4}^1[B_{t_4}^*(\Phi_m \cap U_1)] :$$

$$\begin{align*}
  \sum_{i+j+k+h=m-5} a_{ijkh} x_4^{2i+k+2h} & y_4^{k} z_4^{2h} \\
  + \sum_{i'+j'+k'+h'=m} b_{ijkh} x_4^{2i'+k'+2h'} & y_4^{k'} z_4^{2h'} \varphi_4(x_4^{2i}t_4^{11}, 1, x_4^{11} y_4^{11} t_4^{11}, x_4^{11}z_4^{11} t_4^{11})x_4^{11}t_4^{11} = 0.
\end{align*}$$

From the analogous four claims written above and from the equality

$$\varphi_4(x_4^{2i}t_4^{11}, 1, x_4^{11} y_4^{11} t_4^{11}, x_4^{11}z_4^{11} t_4^{11}) = x_4^{41}(...),$$

we obtain the inequalities

$$\begin{cases} 
  2i + k + 2h + 4 + 1 \geq m \\
  2i' + k' + 2h' \geq m 
\end{cases}, \text{ i.e. } \begin{cases} 
  i + h \geq j \\
  i' + h' \geq j'. 
\end{cases}$$

Let us move on now to consider the two blows-up $\pi_5$ and $\pi_6$. Once again, we can assume that the blow-up $\pi_3$ at $A_2$ is performed first, so we can again use the local blows-up $B_{x_1}, B_{y_2}, B_{z_3}, B_{t_4}$ in section 1.

As in the above cases, here for $\pi_5$ and $\pi_6$ we obtain that the total transform of $\Phi_m \cap U_2$, with respect to $B_{y_2}$, is given by

$$B_{y_2}^*(\Phi_m \cap U_2) : A_{m-5}(x_2y_2, y_2, 1, y_2z_2)\varphi_4(x_2y_2, y_2, 1, y_2z_2) + \psi_m(x_2y_2, y_2, 1, y_2z_2, y_2t_2) = 0.$$
The triple curve $C_2$ infinitely near $A_2$ in affine coordinates $(x_2, y_2, z_2, t_2)$ is given by (section 3) \[
\begin{aligned}
x_2 &= 0 \\
y_2 &= 0 \\
t_2 &= 0
\end{aligned}
\]

The blow-up $\pi_6$ along $C_2$ is locally given by the formulas:
\[
\begin{align*}
B_{x_{21}} : & \begin{cases} 
x_2 = x_{21} \\
y_2 = x_{21} y_{21} \\
z_2 = z_{21} \\
t_2 = x_{21} t_{21}
\end{cases} \\
B_{y_{21}} : & \begin{cases} 
x_2 = x_{22} y_{22} \\
y_2 = y_{22} \\
z_2 = z_{22} \\
t_2 = y_{22} t_{22}
\end{cases} \\
B_{t_{21}} : & \begin{cases} 
x_2 = x_{23} t_{23} \\
y_2 = y_{23} t_{23} \\
z_2 = z_{23} \\
t_2 = t_{23}
\end{cases}
\end{align*}
\]

The total transform of $B_{y_{21}}^* (\Phi_m \cap U_2)$ with respect to $B_{x_{21}}$ is given by
\[
B_{x_{21}}^* [B_{y_{21}}^* (\Phi_m \cap U_2)] = \left( \sum_{i+j+k+h=m-5} a_{ijkh} x_{21}^{2i+j+k+h} y_{21}^i z_{21}^j t_{21}^k \right) \varphi_4 (x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) x_{21}^2 y_{21} t_{21}
+ \sum_{i'+j'+h'+l'=m} b_{ijkh} x_{21}^{2i'+j'+h'+l'} y_{21}^{i'} z_{21}^{j'} t_{21}^{h'} = 0.
\]

From the same four claims written above, and from the equality
\[
\varphi_4 (x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) = x_{21}^2 (\ldots),
\]
we obtain the inequalities
\[
\begin{cases} 
2i + j + k + 2 + 2 \geq m \\
2i' + j' + h' \geq m
\end{cases}
\]
\[
\begin{aligned}
i &\geq k + 1 \\
i' &\geq k'
\end{aligned}
\]

\[\text{\textbullet\textbullet\textbullet}\text{ Finally, considering the two blows-up } \pi_7 \text{ and } \pi_8, \text{ as in the case of } \pi_5 \text{ and } \pi_6, \text{ we obtain the inequalities}
\]
\[
\begin{cases} 
i + 2j + k + 2 + 2 \geq m \\
i' + 2j' + k' \geq m
\end{cases}
\]
\[
\begin{aligned}
i &\geq k + 1 \\
j &\geq h + 1 \\
j' &\geq h'
\end{aligned}
\]

Joining the above inequalities, we obtain
\[
\begin{aligned}
(i + h &\geq j \geq i + 1 \geq k + 2, \\
i' + h' &\geq j' \geq i' \geq k', \\
j' &\geq h')
\end{aligned}
\]

From the inequalities in the first line of (**), we deduce $j \geq 2, i \geq 1, h \geq 1$. Bearing in mind that $i + j + k + h = m - 5$,
\[i) \text{ there are no values of } i,j,k,h \text{ satisfying (**)} \text{ and corresponding to } m, \text{ for } m \leq 8;
\]
\[ii) \text{ the values } [i = 1, j = 2, k = 0, h = 1] \text{ correspond to } m = 9;
\]
iii) there are no values of \(i, j, k, h\) satisfying (***) and corresponding to \(m = 10\);

iv) the two sets of values \([i = 2, j = 3, k = 0, h = 1]\) and \([i = 1, j = 3, k = 0, h = 2]\) satisfy (**) and correspond to \(m = 11\), and so on; there are values of \(i, j, k, h\) satisfying (**) that correspond to any value of \(m \geq 12\).

As for the inequalities in the second line of (**), and given that \(i' + j' + k' + h' = m\),

1) there are no values of \(i', j', k', h'\) satisfying (**) and corresponding to \(m = 1\);
2) the two sets of values \([i' = j' = 1, k' = h' = 0]\) and \([j' = h' = 1, i' = k' = 0]\) satisfy (**) and correspond to \(m = 2\);
3) the two sets of values \([i' = j' = k' = 1, h' = 0]\) and \([i' = j' = h' = 1, k' = 0]\) satisfy (**) and correspond to \(m = 3\);
4) there are 4 sets of values satisfying (**) and corresponding to \(m = 4\), there are also 4 sets of values satisfying (**) and corresponding to \(m = 5\), 8 sets satisfying (**) and corresponding to \(m = 6\) and 8 sets satisfying (**) and corresponding to \(m = 7\).

5) The following sets \([i' = j' = 3, k' = h' = 0]\), \([i' = j' = h' = 2, k' = 0]\), \([i' = j' = 2, k' = h' = 1]\), \([i' = 2, j' = 3, k' = 0, h' = 1]\) are 4 of the 8 sets of values satisfying (**) that correspond to \(m = 6\). The following sets \([i' = j' = 3, k' = 1, h' = 0]\), \([i' = h' = 2, j' = 3, k' = 0]\), \([i' = 1, j' = 3, k' = 1, h' = 2]\), \([i' = 1, j' = h' = 3, k' = 0]\) are 4 of the 8 sets of values satisfying (**) that correspond to \(m = 7\).

Consequences. Let us just recall that we have written the equation of an \(m\)-canonical adjoint \(\Phi_m\) as follows:

\[
\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3) \phi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3, X_4) = 0,
\]

where

\[
A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \left( \sum_{i+j+k+h=m-5} a_{ijkl} X_0^i X_1^j X_2^k X_3^h \right)X_4
\]

and

\[
\psi_m(X_0, X_1, X_2, X_3) = \sum_{i'+j'+h'+l'=m} b_{ijkl} X_0^{i'} X_1^{j'} X_2^{k'} X_3^{h'}.
\]

From i),...,vi), we deduce that the form \(A_{m-5}\) is zero if and only if \(m \leq 8\) and \(m = 10\).
A threefold with \( p_g = 0 \) and \( P_2 = 2 \)

Since the \( m \)-genus \( P_m \) of a desingularization \( X \) of \( V \) is the number of the linearly independent forms defining \( m \)-canonical adjoints to \( V \) (cf. [S1]), from 1), ..., 4), we deduce the following results regarding the plurigenera of a desingularization \( X \) of \( V \).

From 1), we can establish that there are no 1-canonical adjoints (also called canonical adjoints) to \( V \); this implies that the geometric genus of \( X \) is \( p_g = 0 \).

From 2), we find that \( \Phi_2 : \psi_2(X_0, X_1, X_3, X_4) = X_1(\lambda_1 X_0 + \lambda_2 X_3) = 0 \), where \( \lambda_i \in k \); this implies that the bigenus of \( X \) is \( P_2 = 2 \).

From 3), we learn that \( \Phi_3 : \psi_3(X_0, X_1, X_3, X_4) = X_0 X_1(\mu_1 X_2 + \mu_2 X_3) = 0 \), \( \mu_i \in k \); this implies that the trigenus of \( X \) is \( P_3 = 2 \).

From 4), we obtain that \( P_4 = P_5 = 4, P_6 = 8 \) and \( P_7 = 8 \).

In addition, \( X \) has the plurigenera \( P_8 = 13, P_9 = 15, P_{10} = 19, P_{11} = 22 \).

6. The \( m \)-canonical transformation \( \varphi_{\lfloor mK_X \rfloor} \), \( m \geq 2 \).

Let us use \( \alpha_m : V \dashrightarrow \mathbb{P}^{P_m-1} \) to indicate the rational transformation associated with the linear system of \( m \)-canonical adjoints \( \Phi_m \) to \( V \). The following triangle

\[
\begin{array}{c}
\varphi_{\lfloor mK \rfloor} \\
\downarrow \varphi \\
\mathbb{P}^{P_m-1} \\
\downarrow \alpha_m \\
V
\end{array}
\]

is commutative.

Let us consider the linear system of \( m \)-canonical adjoints \( \Phi_m \). From i) and 1), ..., 4) and the Consequences, we can see that if \( 2 \leq m \leq 5 \), then \( \Phi_m \) is given by \( \psi_m(X_0, X_1, X_3, X_4) = 0 \); moreover, the rational transformation \( \alpha_m \) has the generic fiber of dimension \( \geq 1 \). From the commutativity of the above triangle, \( \varphi_{\lfloor mK_X \rfloor} \) also has the generic fiber of dimension \( \geq 1 \).

From i) and 5) and the Consequences, we know that \( \Phi_m \), for \( m = 6, 7 \), is again given by \( \psi_m(X_0, X_1, X_3, X_4) = 0 \), and that the rational transformation \( \alpha_m \), as well as \( \varphi_{\lfloor mK_X \rfloor} \), is generically \( 2 : 1 \). As a consequence of this and of the
fact that $P_2 \neq 0$, $\varphi_{|mK_X|}$ is either generically $2 : 1$ or birational (to its image) for $m \geq 8$. It is not difficult to prove that $\varphi_{|6K_X|}$ and $\varphi_{|7K_X|}$ are generically $2 : 1$, since all we have to do is consider the rational transformation defined by the 4 sets of values given in 5) (in both cases $m = 6, 7$).

Next, we note that a necessary condition for the birationality of $\varphi_{|mK_X|}$ is that $A_{m-5} \neq 0$ in the equation $A_{m-5} \varphi_4 X_4 + \psi_m = 0$ of $\Phi_m$; in other words, $\Phi_m$ must be a non-global canonical adjoint to $V$ (cf. Remark 1, section 5).

To be more precise, let us consider $\Phi_m : A_{m-5} \varphi_4 X_4 + \psi_m = 0$ and assume that the rational transformation $\varphi'_m : V \dasharrow \mathbb{P}^{P_{m-1}}$ defined by the linear system $\psi_m = 0$ of global m-canonical adjoints to $V$ (see Remark 1, section 5) is generically $2 : 1$, then $\varphi_{|mK_X|}$ is birational if and only if $A_{m-5} \neq 0$. This is immediately proved by the presence of the addendum $A_{m-5} X_4$, which contains $X_4$ to the power 1; indeed, this addendum separates the two distinct points on $V : \varphi_4 X_4^2 + \varphi_5 X_4 + \varphi_6 = 0$ that are mapped to one point.

As a corollary of this latter fact, in the light of i),...iv) and the Consequences, $\varphi_{|mK_X|}$ is birational if and only if $m = 9$ and $m \geq 11$. So, for $m = 10$, there is a gap in the birationality of $\varphi_{|mK_X|}$.

This concludes our examination of $\varphi_{|mK_X|}$, for $m \geq 2$.

7. Computing the irregularities of $X$.

This brings us to the demonstration that $q_i = \dim_k H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(X, \mathcal{O}_X) = q(S_r) = \dim_k H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section $S$ of $V$ (cf. [S1], section 4, for instance). $S$ has several isolated (actual or infinitely-near) double points and no other singularities. This follows from the fact that, outside the points $A_0, A_1, A_2, A_3$ and $A_4$, the hypersurface $V$ only has actual or infinitely-near double curves and isolated double points. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36) in section 4 of [S1], which states that:

$$q_2 = p_g(X) + p_g(S_r) - \dim_k(W_2),$$

where $W_2$ is the vector space of the degree 2 forms defining global adjoints $\Phi_2$ to $V$, i.e. defining hyperquadrics $\Phi_2$ such that

$$\pi_2^* \ldots \pi_2^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_9 \geq 0,$$

(cf. the expression of $D_m$ in (*), section 4). So the above hyperquadrics $\Phi_2$
A threefold with \( p_g = 0 \) and \( P_2 = 2 \)

are those passing through the points \( A_0, A_1, A_2, A_3 \) and \( A_4 \). Thus, \( \dim_k(W_2) = 15 - 5 = 10 \). It follows from \( p_g(S_r) = 10 \) and \( p_g(X) = 0 \) (cf. Consequences at the end of section 5) that \( q_2 = 0 \).

REFERENCES


[Ro2] M. C. Ronconi, *Examples of birationality of \( \Phi_{|mK|} \) for large \( m \) with and without gaps*, to appear.


