The Schur Multiplier of a Generalized Baumslag-Solitar Group

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Abstract - The structure of the Schur multiplier of an arbitrary generalized Baumslag-Solitar group is determined and applications to central extensions are described.

1. Introduction and Results.

A generalized Baumslag-Solitar group, or GBS-group, is the fundamental group of a finite connected graph of groups with infinite cyclic vertex and edge groups. In detail let $\Gamma$ be a finite connected graph – multiple edges and loops are allowed – with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For each edge $e$ we choose endpoints $e^+$ and $e^-$, and hence a direction for the edge. Infinite cyclic groups $\langle g_x \rangle$ and $\langle u_e \rangle$ are assigned to each vertex $x$ and edge $e$. Injective homomorphisms $\langle u_e \rangle \to \langle g^{e^+} \rangle$ and $\langle u_e \rangle \to \langle g^{e^-} \rangle$ are defined by the assignments

$$u_e \mapsto g^{\omega^+(e)}_e \quad \text{and} \quad u_e \mapsto g^{\omega^-(e)}_e,$$

where $\omega^+(e), \omega^-(e) \in \mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$. Thus we have a weight function

$$\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*$$

where $\omega(e) = (\omega^-(e), \omega^+(e))$. The weighted graph $(\Gamma, \omega)$ is called a GBS-graph.

The GBS-group determined by the weighted graph $(\Gamma, \omega)$ is the fundamental group

$$G = \pi_1(\Gamma, \omega).$$

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To obtain a presentation of $G$ choose a maximal subtree $T$ of $\Gamma$. Then $G$ has generators

$$g_x, (x \in V(\Gamma)), \text{ and } t_e, \ (e \in E(\Gamma) \setminus E(T)),$$

with defining relations

$$g_{e^+}^{\omega^+(e)} = g_{e^-}^{\omega^-(e)}, \text{ for } e \in E(T),$$

$$(g_{e^+}^{\omega^+(e)} t_e)^{\omega(e)} = g_{e^-}^{\omega^-(e)}, \text{ for } e \in E(\Gamma) \setminus E(T).$$

It is known that up to isomorphism $G$ is independent of the choice of $T$: for this and other basic properties of graphs of groups see [1], [3], [10]. If $\Gamma$ consists of a single loop with weight $(n, m)$, then $\pi_1(\Gamma, \omega)$ is a Baumslag-Solitar group

$$BS(m, n) = \langle t, g \mid (g^m)^t = g^n \rangle.$$  

It is easy to see that $GBS$-groups are torsion-free. They are obviously finitely presented, and in fact every finitely generated subgroup of a $GBS$-group is either free or a $GBS$-group ([6], 2.7: see also [4], 1.2), so such groups are coherent, i.e., all finitely generated subgroups are finitely presented. By an important result of Kropholler ([7]) the non-cyclic $GBS$-groups are exactly the finitely generated groups of cohomological dimension 2 which have an infinite cyclic subgroup commensurable with its conjugates. It is therefore natural to enquire about homology and cohomology of $GBS$-groups in dimensions 1 and 2.

Here we are concerned with integral homology: of course $H_1(G) \cong G_{ab}$, the abelianization, while $H_2(G) = M(G)$ is the Schur multiplier of $G$. Our principal result describes the structure of the Schur multiplier of an arbitrary $GBS$-group.

**Theorem 1.** Let $G$ be a generalized Baumslag-Solitar group. Then $M(G)$ is free abelian of rank $r_0(G) - 1$ where $r_0(G)$ is the torsion-free rank of $G_{ab}$.

**Corollary 1.** The Euler characteristic of a $GBS$-group is 0.

This follows since the homology groups of a $GBS$-group $G$ in dimensions 0, 1, 2 have torsion-free ranks 1, $r_0(G)$, $r_0(G) - 1$ respectively and the alternating sum of these is zero.
We remark that associated with any GBS-group there is a complex $K(\Gamma, \omega)$ defined in [4]. It can be shown that the Euler characteristic of this complex is zero and this observation is the basis for a topological – but not necessarily shorter – treatment of Theorem 1. Details will appear elsewhere ([5]).

**T-dependence.**

The structure of $G_{ab}$, and hence $r_0(G)$, can be found from the abelian presentation of $G_{ab}$ arising from the standard presentation of the GBS-group $G$ by the usual method of Smith normal form. However, this is a lengthy process and, as only $r_0(G)$ is required in order to compute $M(G)$, it is worthwhile to give a simpler method.

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group and let $T$ be the chosen maximal subtree of $\Gamma$. Suppose that $e = \langle x, y \rangle \in E(\Gamma) \setminus E(T)$ where $x \neq y$. Now there is a unique path in the tree $T$ from $x$ to $y$, say $x = x_0, x_1, \ldots, x_n = y$. By reading along this path, we obtain a relation $g_{x_*}^{p_1(e)} = g_y^{p_2(e)}$ where $p_1(e)$ and $p_2(e)$ are the respective products of the left and right weight values of the edges in the path from $x$ to $y$. If the vector $(\omega^-(e), \omega^+(e))$ is a rational multiple of $(p_1(e), p_2(e))$, then $e$ is said to be $T$-dependent, and otherwise $e$ is $T$-independent. If $e$ is a loop, then by convention $p_1(e) = 1 = p_2(e)$ and $e$ is $T$-dependent precisely when $\omega^-(e) = \omega^+(e)$.

The definition of $T$-dependence may be restated as follows.

**Lemma 1.** With the above notation, a non-tree edge $e = \langle x, y \rangle$ of a GBS-graph is $T$-dependent if and only if

$$\frac{\omega^-(e)}{\omega^+(e)} = \frac{p_1(e)}{p_2(e)}.$$

If every non-tree edge of a GBS-graph is $T$-dependent, the GBS-graph is said to be tree-dependent. The torsion-free rank of the abelianization of a GBS-group can be computed from the following result.

**Theorem 2.** Let $G = \pi_1(\Gamma, \omega)$ be a generalized Baumslag-Solitar group. Then

$$r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 + \varepsilon(\Gamma, \omega)$$

where $\varepsilon(\Gamma, \omega) = 1$ if $(\Gamma, \omega)$ is tree-dependent and otherwise equals 0.
(A variant of this result with a different proof appears in [8], Theorem 1.1). We note that, as a consequence of Theorem 2, \( r_0(G) \) can be found by simply inspecting the graph of the GBS-group \( G \). Notice also that \( \varepsilon(\Gamma, \omega) \) depends only on the GBS-graph \( (\Gamma, \omega) \), not on the choice of maximal subtree. Thus the property of tree-dependence is independent of the maximal subtree selected.

We remark that the invariant \( \varepsilon \) is closely related to the centre of a GBS-group and is an important tool in the theory of GBS-groups: it is the subject of an ongoing investigation.

As is well known, knowledge of the structure of the Schur multiplier of a group allows one to draw conclusions about central extensions by the group. As a consequence of Theorem 1 one can determine when all central extensions by a GBS-group \( G \) split, i.e., they are direct products. It is shown in Corollary 4 below that every central extension by a generalized Baumslag-Solitar group \( G \) splits if and only if \( G_{ab} \) is infinite cyclic.

2. Proof of Theorem 2.

Let \( G = \pi_1(\Gamma, \omega) \) be a GBS-group with \( T \) a maximal subtree of \( \Gamma \). Then \( G \) has an abelian presentation with generators \( g_x, t_e \), where \( x \in V(\Gamma), \ e \in E(\Gamma) \setminus E(T) \), subject to the defining relations \( g_x^{\varepsilon(e)} = g^{\omega(e)}_e , \) \( (e \in E(\Gamma)) \). Put \( G_0 = \langle g_x \mid x \in V(\Gamma) \rangle \); then \( G_0 \cong \pi_1(T, \omega) \) and \( r_0(G_0) \leq 1 \) since each pair of generators of \( G_0 \) is linearly dependent. Since \( G_0 \) has fewer relations than generators, it is infinite and \( r_0(G_0) = 1 \). Of course, the stable elements \( t_e \), are linearly independent modulo the torsion subgroup of \( G_{ab} \). Therefore

\[
r_0(G) = |E(\Gamma) \setminus E(T)| + \varepsilon,
\]

where \( \varepsilon = 1 \) if each vertex generator has infinite order modulo \( G' \) and otherwise \( \varepsilon = 0 \). If some non-tree edge \( e \) is \( T \)-independent, then, in the notation of Lemma 1, the relations \( g^{\omega(e)}_e = g^{\omega(e)}_e \) and \( g^{p_1(e)}_e = g^{p_2(e)}_e \) are independent, which forces each vertex generator to have finite order modulo \( G' \); hence \( \varepsilon = 0 \). On the other hand, if all such edges are \( T \)-dependent, i.e., \( (\Gamma, \omega) \) is tree-dependent, then all vertex generators have infinite order and \( \varepsilon = 1 \). Since

\[
|E(\Gamma) \setminus E(T)| = |E(\Gamma)| - (|V(\Gamma)| - 1) = |E(\Gamma)| - |V(\Gamma)| + 1,
\]

the result follows on setting \( \varepsilon(\Gamma, \omega) = \varepsilon \).\qed
3. Proof of Theorem 1.

Let $G = \pi_1(\Gamma, \omega)$ be a GBS-group with $T$ a maximal subtree of $\Gamma$. We recall the following inequality, which is valid for any finitely presented group $H$ with $n$ generators and $r$ relators:

$$n - r \leq r_0(H) - d(M(H)),$$

where $d(X)$ is the minimal number of generators of a group $X$, (see, for example, [9], p.550). In the present situation we have $n = |V(\Gamma)| + |E(\Gamma)\setminus E(T)|$ and $r = |E(\Gamma)|$, so $n - r = 1$. Thus $d(M(G)) \leq r_0(G) - 1$ and it suffices to prove that $r_0(M(G)) \geq r_0(G) - 1$. The proof is by induction on $|E(\Gamma)|$, which may be assumed positive.

(i) We can assume that $r_0(G) > 1$, so that $\Gamma$ is not a tree.

For if $r_0(G) = 1$, then $d(M(G)) = 0$. Note that if $\Gamma$ is a tree, then $r_0(G) = 1$ since each pair of vertex generators is linearly independent.

(ii) Case: $\Gamma$ has a single non-tree edge.

Let $e = \langle x, y \rangle$ be the edge which is not in $T$. Now $r_0(G) \leq 2$ by Theorem 2, so $r_0(G) = 2$ and $\alpha(\Gamma, \omega) = 1$; thus $e$ must be $T$-dependent. Apply the five-term homology sequence for the exact sequence $G' \to G \to G_{ab}$ to get

$$M(G) \to M(G_{ab}) \to G'/[G', G] \to G_{ab} \to G_{ab} \to 1.$$

Note that $r_0(M(G_{ab})) = 1$ since $M(G_{ab}) \approx G_{ab} \wedge G_{ab}$.

We claim that $G'/[G', G]$ is finite. To see this write $t = t_e$ and let $\omega(e) = (h, k)$, so that $(g_y^h)^t = g_x^k$. Also $\langle g_x \rangle \cap \langle g_y \rangle = \langle g_x^m = g_y^n \rangle$ where $m, n \in \mathbb{Z}^*$. By $T$-dependence $(h, k)$ is a rational multiple of $(m, n)$, say $ih = jm$ and $ik = jn$, with $i, j \in \mathbb{Z}^*$. Then

$$[g_y, t]^i \equiv [g_y^i, t] \equiv g_y^{-ik}g_x^i \equiv g_y^{-jn}g_x^jm \equiv 1 \mod [G', G].$$

Next for any vertex generator $g_z$ we have $g_z^r = g_y^s$ for some $r, s \in \mathbb{Z}^*$, and hence

$$[g_z, t]^r \equiv [g_y^r, t] \equiv [g_y, t]^r \mod [G', G].$$

Finally, $[g_u, g_v][G', G]$ has finite order for any vertex generators $g_u, g_v$. It follows that $G'/[G', G]$ is periodic, so it is finite.

Returning to the exact homology sequence above, we conclude that $M(G)$ must be infinite, so that $r_0(M(G)) \geq 1 = r_0(G) - 1$, as required.
(iii) From now on we assume that \( \Gamma \) has at least two non-tree edges.

Let \( e = \langle x, y \rangle \) be one of the non-tree edges and let the unique path in \( T \) from \( x \) to \( y \) be

\[
\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{k-1}, x_k \rangle,
\]

where \( x = x_1 \) and \( y = x_k \). Define subgroups \( G_1 = \langle t_e, g_{x_1}, \ldots, g_{x_k} \rangle \) and

\[
G_2 = \langle t_f, g_x \mid x \in V(\Gamma), f \in E(\Gamma) \setminus E(T), f \neq e \rangle.
\]

Then \( G_i = \pi_1(\Gamma_i, \omega), i = 1, 2 \), where \( \Gamma_1, \Gamma_2 \) are subgraphs of \( \Gamma \) with \( V(\Gamma_1) = \{ x_1, \ldots, x_k \} \), \( V(\Gamma_2) = V(\Gamma) \) and respective edge sets \( \{ e, \langle x_j, x_{j+1} \rangle \mid j = 1, 2, \ldots, k - 1 \} \) and \( E(\Gamma) \setminus \{ e \} \), with restricted weight functions. Furthermore

\[
G = G_1 \ast_U G_2
\]

where \( U = \langle g_{x_1}, g_{x_2}, \ldots, g_{x_k} \rangle \). Since \( U \simeq \pi_1(T_0, \omega) \), with \( T_0 \) the path

\[
\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{k-1}, x_k \rangle,
\]

we have \( r_0(U) = 1 \) and \( M(U) = 0 \) by (i).

Next we form the Mayer-Vietoris sequence for the generalized free product \( G = G_1 \ast_U G_2 \), ([2], p. 51),

\[
\begin{array}{c}
0 = M(U) \rightarrow M(G_1) \oplus M(G_2) \rightarrow M(G) \rightarrow U_{ab} \rightarrow \\
(G_1)_{ab} \oplus (G_2)_{ab} \rightarrow G_{ab} \rightarrow 1.
\end{array}
\]

At this point we must distinguish two cases.

(iv) Case: the graph \( \Gamma \) has a non-tree edge \( e \) which is \( T \)-dependent.

Apply the Mayer-Vietoris sequence above for the edge \( e \). Since \( \Gamma_1 \) has just one non-tree edge \( e \) and it is \( T \)-dependent in \( \Gamma_1 \), we conclude that

\[
r_0(G_1) = 2 \quad \text{and} \quad M(G_1) \simeq \mathbb{Z}
\]

by (ii). Also \( UG_1'/G_1' \) is infinite, so the image of \( (G_1)_{ab} \) in the exact sequence (*) has infinite projection into \( (G)_{ab} \). Therefore

\[
r_0(G) \leq r_0(G_1) + r_0(G_2) - 1 = r_0(G_2) + 1
\]

and \( r_0(G_2) \geq r_0(G) - 1 \). By induction on \( |E(\Gamma)| \) the result is true for \( G_2 \), so we have

\[
r_0(M(G)) \geq r_0(M(G_1) \oplus M(G_2)) \geq 1 + (r_0(G) - 2) = r_0(G) - 1,
\]

as required.

We are now left with the situation:

(v) Case: all non-tree edges in \( \Gamma \) are \( T \)-independent.

Choose any non-tree edge \( e \) and apply the sequence (*) in (iii) for this edge. Since \( e \) is \( T \)-independent, \( r_0(G_1) = 1 \) and \( M(G_1) = 0 \). Also \( UG_1'/G_1' \) is finite because \( e \) is \( T \)-independent. By (iii) there is non-tree edge \( f \neq e \) and
\( t = t_f \in G_2 \). Since \( f \) is \( T \)-independent, \( r_0(\langle t, U \rangle) = 1 \) and also \( U G'_2/G'_2 \) is finite. Consequently the image of \( U_{ab} \) in \((G_1)_{ab} \oplus (G_1)_{ab} \) is finite. Since \( U_{ab} \) is infinite, it follows from the sequence (\( * \)) that the cokernel of the map \( M(G_1) \oplus M(G_2) \rightarrow M(G) \) is infinite.

By induction hypothesis the result holds for \( G_2 \), so we conclude that
\[
\ r_0(M(G)) \geq r_0(M(G_1)) + r_0(M(G_2)) + 1 = (r_0(G_2) - 1) + 1 = r_0(G_2),
\]
since \( M(G_1) = 0 \). Finally, the image of \( U_{ab} \) in the sequence (\( * \)) being finite, we obtain
\[
\ r_0(G) = r_0((G_1)_{ab} \oplus (G_2)_{ab}) = r_0(G_1) + r_0(G_2) = 1 + r_0(G_2).
\]
Hence \( r_0(G_2) = r_0(G) - 1 \) and \( r_0(M(G)) \geq r_0(G) - 1 \), as required. \( \square \)

**Corollary 2.** The GBS-group \( \pi_1(\Gamma, \omega) \) has trivial Schur multiplier if and only if \( \Gamma \) is either a tree or else a tree with one further edge and \( \Gamma \) is not tree-dependent.

**Proof.** By the theorem \( M(G) = 0 \) if and only if \( r_0(G) = 1 \). This condition requires there to be at most one non-tree edge and by Theorem 2 it must be \( T \)-independent. \( \square \)

**Corollary 3.** Every GBS-group has deficiency 1.

**Proof.** Recall that the deficiency \( \text{def}(G) \) of group \( G \) is equal to \( \sup\{n - r\} \) where \( n \) and \( r \) are the respective numbers of generators and relations in an arbitrary finite presentation. If \( G \) is a GBS-group, then \( 1 = \text{def}(G) = r_0(G) - d(M(G)) = 1. \) \( \square \)

**Example.** Consider the GBS-group \( G \) arising from the following GBS-graph,

![GBS Graph](image)

where the maximal subtree chosen is the path \( x, y, z, u \) and the stable elements are \( r, s, t \) as indicated. Then \( G \) has a presentation with generators

\[ r, s, t, g_x, g_y, g_z, g_u \]
and relations

\[(g_x^2)^2 = g_x^2, \quad g_y^2 = g_y^{-3}, \quad g_z^2 = g_z^2, \quad g_u^4 = g_u^1, \quad (g_u^{12})^t = g_y^{20}, \quad (g_x^4)^s = g_y^{-1}.
\]

All non-tree edges with the exception of \(y, x\) are T-dependent. Therefore \((\Gamma, \omega)\) is not tree-dependent, \(e(\Gamma, \omega) = 0\) and \(r_0(G) = |E(\Gamma)| - |V(\Gamma)| + 1 = 3\). Thus \(M(G) \simeq \mathbb{Z} \oplus \mathbb{Z}\).

4. Applications to Central Extensions.

We will now apply our results to yield information about central extensions by GBS-groups. Let \(G\) be a GBS-group and \(C\) an abelian group regarded as a trivial \(G\)-module. Denote by \(F\) the periodic subgroup of \(G_{ab}\); thus \(G_{ab} \simeq F \oplus \mathbb{Z}^{r_0(G)}\) where \(F\) is finite. By the Universal Coefficients Theorem

\[ H^2(G, C) \simeq \text{Ext}(G_{ab}, C) \oplus \text{Hom}(M(G), C) \simeq \text{Ext}(F, C) \oplus \text{Dr} C^{r_0(G)-1}. \]

First we determine when all central extensions of \(C\) by \(G\) are direct products, i.e., when \(H^2(G, C) = 0\).

**Theorem 3.** Let \(G\) be a generalized Baumslag-Solitar group and let \(C \neq 1\) be an abelian group regarded as a trivial \(G\)-module. Then \(H^2(G, C) = 0\) if and only if \(r_0(G) = 1\) and \(C\) is divisible by all primes \(p \in \pi(G_{ab})\).

**Proof.** With the notation used above, \(H^2(G, C) = 0\) if and only if \(\text{Ext}(F, C) = 0\) and \(r_0(G) = 1\). Since \(F\) is finite and \(\text{Ext}(\mathbb{Z}_n, C) \simeq C/C^n\), it follows that \(\text{Ext}(F, C) = 0\) if and only if \(C = C^p\) for all \(p \in \pi(G_{ab})\). (For the elementary properties of \(\text{Ext}\) used here see [9], 7.2).

**Corollary 4.** The following conditions on a generalized Baumslag-Solitar group \(G\) are equivalent

(i) \(H^2(G, \mathbb{Z}) = 0\);
(ii) \(G_{ab} \simeq \mathbb{Z}\);
(iii) \(H^2(G, C) = 0\) for all abelian groups \(C\).

**Proof.** Clearly condition (i) implies that \(\pi(G_{ab})\) is empty and so (ii) holds. Also (ii) implies (iii), while trivially (iii) implies (i).
For example, if \( G = BS(m, n) \), then \( G_{ab} \simeq \mathbb{Z} \oplus \mathbb{Z}_{|m-n|} \), so that \( G \) has the property of Corollary 4 if and only if \( |m - n| = 1 \).

There are corresponding results for homology, which can be proved in an analogous way by using the Universal Coefficients Theorem for homology,

\[ H_2(G, C) \simeq \Tor(G_{ab}, C) \oplus (M(G) \otimes C), \]

and elementary properties of \( \Tor \), (see [9], 7.1).

**Theorem 4.** Let \( G \) be a generalized Baumslag-Solitar group and let \( C \neq 1 \) an abelian group regarded as a trivial \( G \)-module. Then \( H_2(G, C) = 0 \) if and only if \( r_0(G) = 1 \) and \( C_p = 1 \) for all primes \( p \in \pi(G_{ab}) \).

**Corollary 5.** The following conditions on a generalized Baumslag-Solitar group \( G \) are equivalent:

(i) \( H_2(G, \mathbb{Q}/\mathbb{Z}) = 0 \);
(ii) \( G_{ab} \simeq \mathbb{Z} \);
(iii) \( H_2(G, C) = 0 \) for all abelian groups \( C \).

**References**


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