A categorification of quantum sl(n)

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Abstract. To an arbitrary root datum we associate a 2-category. For root datum corresponding to sl(n) we show that this 2-category categorifies the idempotented form of the quantum enveloping algebra.

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1. Introduction

In this paper we categorify the Beilinson–Lusztig–MacPherson idempotented modification $\mathcal{U}(\mathfrak{sl}_n)$ of $U_q(\mathfrak{sl}_n)$ for any $n$, generalizing [21], [22], where such categorification was described for $n = 2$, and using constructions and results of [16], [17] which contain a categorification of $\mathfrak{U}^{-}$ for any Cartan datum. More generally, we define a 2-category associated to any root datum. The categorification of $\mathcal{U}(\mathfrak{sl}_n)$ is given by the 2-category associated to the root system $A_{n-1}$.

In [24] Lusztig associates a quantum group $\mathcal{U}$ to any root datum; the latter consists of a perfect pairing $(.)$ between two free abelian groups $X$ and $Y$, embeddings of the set $I$ of simple roots into $X$, $Y$, and a bilinear form on $\mathbb{Z}[I]$ subject to certain compatibility and integrality conditions. Lusztig’s definition is slightly different from the original ones due to Drinfeld [9] and Jimbo [13]. Lusztig then modifies $\mathcal{U}$ to the nonunital ring $\hat{\mathcal{U}}$ which contains a system of idempotents $\{1_\lambda\}$ over all weights $\lambda \in X$ as a substitute for the unit element,

$$\hat{\mathcal{U}} = \bigoplus_{\lambda, \mu \in X} 1_\mu \hat{\mathcal{U}} 1_\lambda.$$

In the $\mathfrak{sl}_n$ case $\hat{\mathcal{U}}$ was originally defined by Beilinson, Lusztig, and Macpherson [2] and then appeared in [23], [14] in greater generality. It is clear from Lusztig’s work that the $\mathfrak{U}(q)$-algebra $\mathcal{U}$ is natural for at least the following reasons:

1. A $\hat{\mathcal{U}}$-module is the same as a $\mathcal{U}$-module which has an integral weight decomposition. These modules are of prime importance in the representation theory of $\mathcal{U}$.

2. $\hat{\mathcal{U}}$ has analogues of the comultiplication, the antipode, and other standard symmetries of $\mathcal{U}$.

3. $\hat{\mathcal{U}}$ is a $\mathcal{U}$-bimodule.

4. The Peter–Weyl theorem and the theory of cells can be intrinsically stated in terms of the algebra $\hat{\mathcal{U}}$.

5. $\hat{\mathcal{U}}$ has an integral form $\mathfrak{A}\hat{\mathcal{U}}$, a $\mathbb{Z}[q, q^{-1}]$-lattice closed under multiplication and comultiplication. The integral form comes with a canonical basis $\hat{\mathfrak{B}}$. Conjecturally, multiplication and comultiplication in this basis have coefficients in $\mathbb{N}[q, q^{-1}]$ when the Cartan datum is symmetric.

6. The braid group associated to the Cartan datum acts on $\hat{\mathcal{U}}$.

Moreover, $\hat{\mathcal{U}}$ appears throughout the categorification program for quantum groups. Representations of quantum groups that are known to have categorifications all have
integral weight decompositions, and thus automatically extend to representations of $\mathbf{U}$. In most or all of these examples, see [3], [7], [10], [31], [33], the weight decomposition of representations lifts to a direct sum decomposition of categories, so one obtains a categorification of the idempotent $1_\lambda$ as the functor of projection onto the corresponding direct summand (the only possible exception is the categorification of tensor products via the affine Grassmannian [5], [6]). In the categorification of tensor powers of the fundamental $U_q(\mathfrak{sl}_2)$-representation [3], [10], each canonical basis element of $\mathbf{U}$ acts as an indecomposable projective functor or as the zero functor. The idea that $\mathbf{U}$ rather than $\mathbf{U}$ should be categorified goes back to Crane and Frenkel [8].

$\mathbf{U}$ is generated by elements $E_i 1_\lambda$, $F_i 1_\lambda$, and $1_\lambda$, where $\lambda \in X$ is an element of the weight lattice and $i$ is a simple root. We will often write $E_{+,i}$ instead of $E_i$ and $E_{-,i}$ instead of $F_i$. We have

$$E_{\pm i} 1_\lambda = 1_{\mu} E_{\pm i} 1_\lambda,$$

where, in our notations, explained in Section 2.1, $\mu = \lambda \pm i_X$, and $i_X$ is the element of $X$ associated to the simple root $i$. Algebra $\mathbf{U}$ is spanned by products

$$E_{i_{1}} 1_\lambda := E_{\pm i_{1}} E_{\pm i_{2}} \ldots E_{\pm i_{m}} 1_\lambda = 1_{\mu} E_{\pm i_{1}} E_{\pm i_{2}} \ldots E_{\pm i_{m}} 1_\lambda,$$

where $i = (\pm i_{1}, \ldots, \pm i_{m})$ is a signed sequence of simple roots, and $\mu = \lambda + i_X$.

The integral form $\mathbf{A} \mathbf{U} \subset \mathbf{U}$ is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by divided powers

$$E_{i^{(a)}} 1_\lambda = \frac{1}{[a]!} E_{i}^a 1_\lambda.$$

Note that $\mathbf{U}$ can, alternatively, be viewed as a pre-additive category with objects $\lambda \in X$ and morphisms from $\lambda$ to $\mu$ being $1_{\mu} \mathbf{U} 1_\lambda$. Of course, any ring with a collection of mutually orthogonal idempotents as a substitute for the unit element can be viewed as a pre-additive category and vice versa. From this perspective, though, we can expect the categorification of $\mathbf{U}$ to be a 2-category.

In Section 3.1 we associate a 2-category $\mathbf{U}$ to a root datum. The objects of this 2-category are integral weights $\lambda \in X$, the morphisms from $\lambda$ to $\mu$ are finite formal sums of symbols $E_i 1_\lambda \{ t \}$, where $t = (\pm i_{1}, \ldots, \pm i_{m})$ is a signed sequence of simple roots such that the left weight of the symbol is $\mu$ ($E_i 1_\lambda = 1_{\mu} E_i 1_\lambda$), and $t \in \mathbb{Z}$ is a grading shift. When $i$ consists of a single term, we get 1-morphisms $E_{i} 1_\lambda$ and $E_{-i} 1_\lambda$, which should be thought of as categorifying elements $E_i 1_\lambda$ and $F_i 1_\lambda$ of $\mathbf{U}$, respectively. Grading shift $\{ t \}$ categorifies multiplication by $q^t$. The 1-morphism $E_i 1_\lambda : \lambda \rightarrow \mu$ should be thought of as a categorification of the element $E_i 1_\lambda$. When the sequence $i$ is empty, we get the identity morphism $1_\lambda : \lambda \rightarrow \lambda$, a categorification of the element $1_\lambda$.

Two-morphisms between $E_i 1_\lambda \{ t \}$ and $E_j 1_\lambda \{ t' \}$ are given by linear combinations of degree $t - t'$ diagrams drawn on the strip $\mathbb{R} \times [0, 1]$ of the plane. The diagrams consist of immersed oriented one-manifolds, with every component labelled by a
simple root, and dots placed on the components. Labels and orientations at the lower and upper endpoints of the one-manifold must match the sequences $i$ and $j$, respectively. Integral weights label regions of the plane cut out by the one-manifold, with the rightmost region labelled $\lambda$. Each diagram has an integer degree assigned to it. We work over a ground field $k$, and define a 2-morphism between $\mathcal{E}_i \mathbf{1}_\lambda \{t\}$ and $\mathcal{E}_j \mathbf{1}_\lambda \{t'\}$ as a linear combination of such diagrams of degree $t - t'$, with coefficients in $k$, modulo isotopies and a collection of very carefully chosen local relations. The set of 2-morphisms $\mathcal{U}(\mathcal{E}_i \mathbf{1}_\lambda \{t\}, \mathcal{E}_j \mathbf{1}_\lambda \{t'\})$ is a $k$-vector space. We also form graded vector space
\[
\text{HOM}_\mathcal{U}(\mathcal{E}_i \mathbf{1}_\lambda, \mathcal{E}_j \mathbf{1}_\lambda) := \bigoplus_{t \in \mathbb{Z}} \mathcal{U}(\mathcal{E}_i \mathbf{1}_\lambda \{t\}, \mathcal{E}_j \mathbf{1}_\lambda).
\]

Vertical composition of 2-morphisms is given by concatenation of diagrams, horizontal composition consists of placing diagrams next to each other.

In each graded $k$-vector space $\text{HOM}_\mathcal{U}(\mathcal{E}_i \mathbf{1}_\lambda, \mathcal{E}_j \mathbf{1}_\lambda)$ we construct a homogeneous spanning set $B_{i,j,\lambda}$ which depends on extra choices. The Laurent power series in $q$, with the coefficient at $q^r$ equal to the number of spanning set elements of degree $r$, is proportional to suitably normalized inner product $\langle E_i 1_\lambda, E_j 1_\lambda \rangle$, where the semilinear form $\langle \cdot, \cdot \rangle$ is a mild modification of the Lusztig bilinear form on $\mathcal{U}$. The proportionality coefficient $\pi$ depends only on the root datum.

We say that our graphical calculus is non-degenerate for a given root datum and field $k$ if for each $i, j, \lambda$ the homogeneous spanning set $B_{i,j,\lambda}$ is a basis of the $k$-vector space $\text{HOM}_\mathcal{U}(\mathcal{E}_i \mathbf{1}_\lambda, \mathcal{E}_j \mathbf{1}_\lambda)$. Nondegeneracy will be crucial for our categorification constructions.

The 2-category $\mathcal{U}$ is $k$-additive, and we form its Karoubian envelope $\hat{\mathcal{U}}$, the smallest 2-category which contains $\mathcal{U}$ and has splitting idempotents. Namely, for each $\lambda, \mu \in X$, the category $\mathcal{U}(\lambda, \mu)$ of morphisms $\lambda \to \mu$ is defined as the Karoubian envelope of the additive $k$-linear category $\mathcal{U}(\lambda, \mu)$. The split Grothendieck category $K_0(\mathcal{U})$ is a pre-additive category with objects $\lambda$, and the abelian group of morphisms from $\lambda$ to $\mu$ is the split Grothendieck group $K_0(\mathcal{U}(\lambda, \mu))$ of the additive category $\mathcal{U}(\lambda, \mu)$. The grading shift functor on $\mathcal{U}(\lambda, \mu)$ turns $K_0(\mathcal{U}(\lambda, \mu))$ into a $\mathbb{Z}[q, q^{-1}]$-module. This module is free with the basis given by isomorphism classes of indecomposable objects of $\mathcal{U}(\lambda, \mu)$, up to grading shifts. The split Grothendieck category $K_0(\mathcal{U})$ can also be viewed as a nonunital $\mathbb{Z}[q, q^{-1}]$-algebra with a collection of idempotents $[I_\lambda]$ as a substitute for the unit element.

In Section 3.6 we set up a $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism
\[
y : \hat{\mathcal{U}} \to K_0(\mathcal{U})
\]
which takes $1_\lambda$ to $[I_\lambda]$ and $E_i 1_\lambda$ to $[\mathcal{E}_i 1_\lambda]$, for any “divided power” signed sequence $i$.

The main results of this paper are the following theorems.

**Theorem 1.1.** The map $y$ is surjective for any root datum and field $k$. 

Theorem 1.2. The map $\gamma$ is injective if the graphical calculus for the root datum and field $k$ is non-degenerate.

Theorem 1.3. The graphical calculus is non-degenerate for the root datum of $\mathfrak{sl}_n$ and any field $k$.

The three theorems together immediately imply

Proposition 1.4. The map $\gamma$ is an isomorphism for the root datum of $\mathfrak{sl}_n$ and any field $k$.

The last result establishes a canonical isomorphism

$$ \mathcal{A}\mathcal{U}(\mathfrak{sl}_n) \cong K_0(\mathcal{U}(\mathfrak{sl}_n)) $$

and allows us to view $\mathcal{U}(\mathfrak{sl}_n)$ as a categorification of $\mathcal{U}(\mathfrak{sl}_n)$.

Theorem 1.1, proved in Section 3.8, follows from the results of [16], [17], [21] and basic properties of Grothendieck groups and idempotents. Theorem 1.2, proved in Section 3.9, follows from the non-degeneracy of the semilinear form on $\mathcal{U}$ and its pictorial interpretation explained in Section 2.2. To prove theorem 1.3 we construct a family of 2-representations of $\mathcal{U}$ and check that the elements of each spanning set $B_{i,j,\lambda}$ act linearly independently on vector spaces in these 2-representations, implying non-degeneracy of the graphical calculus. Sections 4-6 are devoted to these constructions.

Indecomposable 1-morphisms, up to isomorphism and grading shifts, constitute a basis of $K_0(\mathcal{U}(\mathfrak{sl}_n)) \cong \mathcal{A}\mathcal{U}(\mathfrak{sl}_n)$, which might potentially depend on the ground field $k$. The multiplication in this basis has coefficients in $\mathbb{N}[q,q^{-1}]$. It is an open problem whether this basis coincides with the Lusztig canonical basis of $\mathcal{A}\mathcal{U}(\mathfrak{sl}_n)$. The answer is positive when $n = 2$, see [21].

Another major problem is to determine for which root data the graphical calculus is non-degenerate. Nondegeneracy immediately implies, via Theorems 1.1 and 1.2, that $\mathcal{U}$ categorifies $\hat{\mathfrak{u}}$ for a given root datum.

We believe that $\mathcal{U}$ will prove ubiquitous in representation theory. This 2-category or its mild modifications is expected to act on parabolic-singular blocks of highest weight categories for $\mathfrak{sl}_N$ in the context of categorification of $\mathfrak{sl}_n$ representations [7], [10], [31], on derived categories of coherent sheaves on Kronheimer–Nakajima [20] and Nakajima [26] quiver varieties and on their Fukaya–Floer counterparts, on categories of modules over cyclotomic Hecke and degenerate Hecke algebras [1], [19], on categories of perverse sheaves in Zheng’s categorifications of tensor products [33], on categories of modules over cyclotomic quotients of rings $R(\nu)$ in [16], [17], on categories of matrix factorizations that appear in [18, Section 11], etc. A possible approach to proving that the calculus is non-degenerate for other root systems is to show that $\mathcal{U}$ acts on a sufficiently large 2-category and verify that the spanning
set elements act linearly independently. It would also be interesting to relate our constructions with those of Rouquier [29], [30].

Categories of projective modules over rings $R(v)$, defined in [16], [17], categorify $U^-$ weight spaces. A subset of our defining local relations on 2-morphisms gives the relations for rings $R(v)$. This subset consists exactly of the relations whose diagrams have no critical points (U-turns) on strands and have all strand orientations going in the same direction. In other words, the relations on braid-like diagrams allow us to categorify $U^-$, while the relations without these restrictions lead to a categorification of the entire $U$, at least in the $\mathfrak{sl}_n$ case. Informally, the passage from a categorification of $U^-$ to a categorification of $P\, U$ is analogous to generalizing from braids to tangles.

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2. Graphical interpretation of the bilinear form

2.1. Quantum groups

2.1.1. Algebras $f$ and $U$. We recall several definitions, following [24]. A Cartan datum $(I, \cdot)$ consists of a finite set $I$ and a symmetric $\mathbb{Z}$-valued bilinear form on $\mathbb{Z}[I]$, subject to conditions

- $i \cdot i \in \{2, 4, 6, \ldots \}$ for $i \in I$,
- $d_{ij} := -2\frac{i+j}{\gcd(i,j)} \in \{0, 1, 2, \ldots \}$ for any $i \neq j$ in $I$.

Let $q_i = q_{\frac{i}{2}}$, $[a]_i = q_i^{a-1} + q_i^{a-3} + \cdots + q_i^{1-a} [a]_i! = [a]! [a-1]! \ldots [1]!$. Denote by $f$ the free associative algebra over $\mathbb{Q}(q)$ with generators $\theta_i$, $i \in I$, and introduce $q$-divided powers $\theta_i^{[a]} = \theta_i^a/ [a]!$. The algebra $f$ is $\mathbb{N}[I]$-graded, with $\theta_i$ in degree $i$. The tensor square $f \otimes f$ is an associative algebra with twisted multiplication

$$(x_1 \otimes x_2)(x'_1 \otimes x'_2) = q_{-|x_2||x'_1|}x_1 x'_1 \otimes x_2 x'_2$$

for homogeneous $x_1, x_2, x'_1, x'_2$. The assignment $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ extends to a unique algebra homomorphism $r : f \rightarrow f \otimes f$.

The algebra $f$ carries a $\mathbb{Q}(q)$-bilinear form determined by the conditions

- $(1, 1) = 1$,
- $(\theta_i, \theta_j) = \delta_{i,j} (1 - q_i^2)^{-1}$ for $i, j \in I$,
- $(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in f$,
- $(xx', y) = (x \otimes x', r(y))$ for $x, x', y \in f$.

The bilinear form $(\ , \ )$ is symmetric. Its radical $3$ is a two-sided ideal of $f$. The form $(\ , \ )$ descends to a non-degenerate form on the associative $\mathbb{Q}(q)$-algebra $f = f/\mathfrak{3}$.

\[\text{\footnote{Our bilinear form $(\ , \ )$ corresponds to Lusztig’s bilinear form $\{ \ , \ \}$, see Lusztig [24, 1.2.10].}}\]