Abstract. Explicit calculations play an important role in the theoretical development of the cohomology of groups and its applications. It is becoming more common for such calculations to be derived with the aid of a computer. This mini-workshop assembled together experts on a diverse range of computational techniques relevant to calculations in the cohomology of arithmetic groups and applications in algebraic K-theory and number theory with a view to extending the scope of computer aided calculations in this area.


Introduction by the Organisers

The mini-workshop Computations in the Cohomology of Arithmetic Groups was attended by 16 participants from 7 different countries with various expertise on the topics of the workshop. The week was organized around 14 short talks, a series of 3 talks given by Professor Günter Harder, a software session presenting new tools which can be used for the explicit calculations in the cohomology of arithmetic groups and related topics, and a problem session (presented at the end of this report). The schedule also included large time periods for discussions and collaborations. The speakers presented survey talks and new results including related problems. Some questions have been even addressed during the week (see for instance the abstract of J. Lannes related to a problem of G. Harder).

The cohomology of arithmetic groups is a rich subject with links to geometry, topology, ring theory and number theory. A classic example of such a group is the
general linear group $GL_N(\mathbb{Z})$ over the integers. A theorem of Borel is that the rational homology of this group in degree $d$ does not depend on $N$ for sufficiently large $N$. Moreover, Borel explicitly computed the homology in these cases, called the stable range. If one replaces $GL_N(\mathbb{Z})$ by a congruence subgroup, then Borel’s theorem still applies. Machine computations for congruence subgroups in the unstable range are being pursued by a number of independent research groups across Europe and the US. One motivation for such computations is a theorem of Franke that establishes a deep connection between the cohomology of congruence subgroups $\Gamma_0$ and the study of automorphic forms: the cohomology $H^*(\Gamma_0; M)$ with suitable coefficients can be contracted from certain automorphic forms, namely those of homological type. The cohomology can be used to test various number theoretic conjectures, in particular those concerning Hecke operators on $H^*(\Gamma_0; M)$. Work of Quillen, Charney, and van der Kallen implies that the integral cohomology of $GL_N(O_K)$ in degree $d$ (for $O_K$ the ring of integers of a number field) is also independent of $N$ for large $N$. In this case, the cohomology groups are intimately related to the algebraic $K$-theory of $O_K$. Recent computational work has yielded the algebraic $K$-group $K_4(\mathbb{Z})$. While the algebraic $K$-theory of number rings can be deduced in large parts from the Bloch/Kato conjectures, several groups are still out of reach, such as the $K_4(\mathbb{Z})$ related to the Kummer/Vandiver conjecture, and it is still difficult to compute explicit $K$-theory classes or their associated regulators. Furthermore, knowledge on the algebraic $K$-theory cannot descend to the cohomology of the related arithmetic groups and even low dimensional homology groups of arithmetic groups are not fully understood.

On one hand, we would like to be able to test conjectures for higher rank arithmetic groups (including symplectic groups) or give explicit evidence of cohomological classes. On the other hand, we would like to be able to compute explicitly, even using advanced machine calculations on state-of-the-art computers, the full structure of the cohomology groups (possibly even their associated cohomology rings) or at least the “less trivial part”. Those computations involve also numerous computations on the cohomology of finite groups, on which there has been much recent progress via their related topological and geometric models.

The recent (and on-going) works presented during this workshop aimed at initiating immediate progress on the above problems, addressing either theoretical or computational aspects, thereby setting the stage for new collaborations after the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.
Mini-Workshop: Computations in the Cohomology of Arithmetic Groups

Table of Contents

Renaud Coulangeon (joint with Oliver Braun, Gabriele Nebe and Sebastian Schönnenbeck)
Computing units in maximal orders using the Voronoï algorithm ........ 2945

Rob de Jeu (joint with David Burns, Herbert Gangl, Alexander Rahm and Dan Yasaki)
Tessellations, Bloch groups, homology groups ......................... 2945

Mathieu Dutour-Sikirić
Lattices and perfect form theory for cohomology computations ........ 2946

Graham Ellis (joint with Bui Anh Tuan, Mathieu Dutour-Sikirić and Le Van Luyen)
Computing cohomology of groups and 2-types .......................... 2949

David J. Green (joint with Simon King)
Computing the cohomology of $p$-groups ............................... 2950

Paul E. Gunnells
Modular symbols and the Sharbly complex .............................. 2951

Günter Harder
Explicit computation of denominators of Eisenstein cohomology classes . 2954

Hans-Werner Henn
Calculating the cohomology of discrete groups via centralizer approximations ...................................................... 2956

Kevin Hutchinson
On the low-dimensional homology of $SL_2$ of $S$-integers ............... 2958

Jean Lannes (joint with Gaëtan Chenevier)
Harder's congruences and related computations ......................... 2959

Gabriele Nebe (joint with Markus Kirschmer)
Construction of discrete chamber transitive actions on affine buildings . 2961

Alexander D. Rahm (joint with Anh Tuan Bui and Matthias Wendt)
A new algorithm for computing Farrell–Tate and Bredon homology for arithmetic groups .............................................. 2962

Sebastian Schönnenbeck (joint with Alex Bocharov and Vadym Kliuchnikov)
$S$-arithmetic groups in quantum computing ............................. 2965
Dan Yasaki

*Computation of certain modular forms using Voronoi polytopes* ........ 2967

Problem session

*Open problems in the cohomology of arithmetic groups* ................. 2970