Definability and Decidability Problems in Number Theory

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Abstract. This workshop brought together experts working on variations
of Hilbert’s Tenth Problem and more general decidability issues for struc-
tures other than the ring of integers arising naturally in number theory and
algebraic geometry.

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Introduction by the Organisers

This highly interdisciplinary workshop brought together 51 Mathematicians from
Number Theory, Logic, Algebraic Geometry, Computability, Model Theory, Arithmetic of Fields, Valuation Theory, and some other related areas. Many contributions and discussions were inspired and driven by the big open decidability ques-
tions such as Hilbert’s Tenth Problem over \( \mathbb{Q} \), the decidability of the first-order
theory of \( \mathbb{F}_p((t)) \) or of \( \mathbb{C}(t) \), variations of Büchi’s Problem and other weak forms
of arithmetic, as well as associated questions of definability and logical complexity
in various rings of number theoretic interest, and in analogous rings of functions.
Several of these issues are closely related to major conjectures in Arithmetic Ge-
ometry, thus faring in deep waters. However, what was most remarkable about
the workshop was the immense and effective effort the participants made in be-
ing understood, in getting across their key points, and in promoting the common
understanding. There was an open, friendly yet well-focused atmosphere, a high
spirit of joint venture, possibly propelled by the dynamics between the large num-
bers of both excellent young researchers on the one hand, and the rather matured
experts on the other. And, of course, the wonderful setting of MFO, and the extreme degree of professionality it is run by on all levels, have played a crucial part in making this workshop such a success.

Let us briefly mention some of the scientific highlights (not including the more survey-like contributions by Colliot-Thélène, Fehm and Derakhshan). One of them was Philip Dittmann’s theorem that irreducibility is diophantine, i.e. definable by an existential first-order formula, in global fields. This vastly generalises partial earlier results in this direction by Poonen, Koenigsmann, Park, Colliot-Thélène and Van Geel. In the case of $\mathbb{Q}$, this theorem (which now holds unconditionally) would follow if $\mathbb{Z}$ was diophantine in $\mathbb{Q}$ (which is one of the big open problems in the field, and which would imply that Hilbert’s Tenth Problem for $\mathbb{Q}$ is unsolvable).

Another breakthrough towards proving that $\mathbb{Z}$ is not diophantine in $\mathbb{Q}$ (and $\mathbb{F}_p[t]$ not in $\mathbb{F}_p(t)$ etc.) is Hector Pasten’s theorem that these negative results follow from a new conjecture of his on the behavior of proximity functions in diophantine approximation – a conjecture that he has verified in a number of cases. He also discussed recent work with Ram Murty showing that standard analytic conjectures on $L$-functions imply that $\mathbb{Z}$ is diophantine in $\mathcal{O}_K$ for all number fields $K$, nicely complementing similar results of Mazur and Rubin which assume conjectures on $\mathcal{X}$.

Natalia Garcia-Fritz developed a powerful machinery generalising Vojta’s approach for solving Büchi’s problem (modulo Bombieri-Lang) by finding all curves of low genus in surfaces with unconditional arithmetic applications à la Büchi for function fields of characteristic zero.

There were two undecidability results for certain infinite extensions of global fields, one by Kirsten Eisenträger for the perfect closure of a global field of positive characteristic, and one by Martin Widmer for sufficiently ramified extensions of $\mathbb{Q}$ following the track set out by Videla and Vidaux along the lines of Julia Robinson’s classical undecidability result for the ring of totally real integers.

By contrast, we had one contribution in the opposite direction (towards decidability) in Győry’s talk, where he described his work (and of his collaborators) on effective finiteness results for certain diophantine equations over $\mathbb{Z}$ and over finitely generated domains, vastly extending Baker’s results from the 1960s on integral points on certain curves.

There were two major contributions from Model Theory: Itay Kaplan showed that the structure $(\mathbb{Z}, +, \mathbb{P}')$ is decidable (where $\mathbb{P}' = \{\pm p \mid p \in \mathbb{P}\}$ with $\mathbb{P}$ denoting the set of rational primes) and is of $U$-rank 1, while $(\mathbb{N}, +, \mathbb{P})$ is known to be undecidable (both under Dickson’s conjecture). And Thomas Scanlon answered two questions about the logical complexity of finitely generated commutative rings proving, firstly, that for any such ring $R$, there is a first-order sentence in the language of rings that determines $R$ (among the f.g. comm. rings) up to isomorphism and, secondly, characterising those infinite f.g. rings that are biinterpretable with
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$\mathbb{Z}$ (e.g. all infinite finitely generated integral domains are). This nicely complemented Florian Pop’s talk addressing the long-standing question of whether all f.g. fields are, up to isomorphism, determined by their first-order theory.

Finally, let us highlight two important valuation theoretic inputs which were similar to each other in providing a new conceptual framework for dealing with decidability issues for valued fields, one by Raf Cluckers about “Resplendent Minimality”, a notion analogous to o-minimality but tailored towards the analysis of henselian valued fields, the idea being that definable subsets of the line are controlled by a finite set of points (like the end points of intervals in the o-minimal setting). The other was Franz-Viktor Kuhlmann’s report on his and others’ research on the new notion of extremal valued fields, that is, valued fields in which for each polynomial (in several variables) the set of values obtained by plugging in elements of the valuation ring always attains a maximum. It is easy to check that $\mathbb{F}_p((t))$ is extremal, and the hope is that this property (together with residue field $\mathbb{F}_p$ and value group elementarily equivalent to $\mathbb{Z}$) suffices to axiomatise $\mathbb{F}_p((t))$ (thus proving its decidability).

Thursday afternoon was almost entirely devoted to a very interesting “Open Problems” session, chaired by Jeroen Demeyer, for which we give a separate “extended abstract”.

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Workshop: Definability and Decidability Problems in Number Theory

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