Mini-Workshop: Applied Koopmanism

Organised by
Didier Henrion, Toulouse
Igor Mezić, Santa Barbara
Mihai Putinar, Santa Barbara

7 February – 13 February 2016

Abstract. Koopman and Perron–Frobenius operators are linear operators that encapsulate dynamics of nonlinear dynamical systems without loss of information. This is accomplished by embedding the dynamics into a larger infinite-dimensional space where the focus of study is shifted from trajectory curves to measurement functions evaluated along trajectories and densities of trajectories evolving in time. Operator-theoretic approach to dynamics shares many features with an optimization technique: the Lasserre moment–sums-of-squares (SOS) hierarchies, which was developed for numerically solving non-convex optimization problems with semialgebraic data. This technique embeds the optimization problem into a larger primal semidefinite programming (SDP) problem consisting of measure optimization over the set of globally optimal solutions, where measures are manipulated through their truncated moment sequences. The dual SDP problem uses SOS representations to certify bounds on the global optimum. This workshop highlighted the common threads between the operator-theoretic dynamical systems and moment–SOS hierarchies in optimization and explored the future directions where the synergy of the two techniques could yield results in fluid dynamics, control theory, optimization, and spectral theory.

Mathematics Subject Classification (2010): 37M25, 90C22, 93B28.

Introduction by the Organisers

In a remarkable outline of mathematics and its future at the beginning of XX-th century, Poincaré suggested that complicated dynamics governed by non-linear partial differential equations can be reduced to and analyzed with the novel (at that time) linear infinite dimensional spectral methods advocated by Hilbert and Fredholm. [28]. Prompted by the advance of spectral theory for unitary and unbounded
self-adjoint operators, Poincaré’s vision became reality a good two decades later by landmark contributions of Carleman [7], Koopman [11] and von Neumann [27]. Although originally aimed at ergodic theory, these linear operator reductions of dynamical systems have far reaching implications and a much wider, unexpected area of applicability. The modern high potential of computer simulations and accumulation of big data have imposed a reconsideration and full appreciation of Poincaré’s bold prophecy. It suffices to note the recent surge of interest for Carleman linearization, spectral analysis of Koopman’s operator or Koopman-von Neumann mechanics. From our narrow perspective, we witness today a proliferation of results utilizing this very operator-theoretic approach in the study of dynamical and control systems [5][12][22]. The need of integration of apparently disparate efforts into a comprehensive theory was the principal motivation of the mini-workshop. The main focus was the interplay between ergodic theory, operator theory, geometric dynamical systems and convex optimization methods.

Two classes of linear operators were of particular interest for the workshop: Koopman-type operators [5] and Perron–Frobenius-type operators [8][12]. Koopman (or composition) operator is a linear infinite-dimensional operator that can be defined for any nonlinear dynamical system. The linear operator retains the full information of the nonlinear state-space dynamics. The formalism based on Koopman operator representation holds promise for extension of dynamical systems methods to systems in high-dimensional spaces as well as hybrid systems, with a mix of smooth and discontinuous dynamics. Recently, Koopman operator properties have been intensely studied, and applications pursued in fields as diverse as fluid mechanics and power grid dynamics. Perron–Frobenius operator is also a linear operator, and, when defined in an appropriate function space, the adjoint to the Koopman operator. Physically, the Perron–Frobenius operator is useful in studying propagation of dynamical systems’ densities. It has shown major promise for applications such as Lagrangian properties of fluid flows and control and optimization of dynamical systems. Next we describe the main themes of the workshop.

One of the topics that indicates how merging of techniques from optimization and ergodic theory can be useful is the development of dedicated convex optimization techniques for the numerical study of dynamical systems. More specifically, we are interested in tailoring the moment-SOS hierarchies of semidefinite programming (SDP) – originally developed for polynomial optimization – to obtain relevant information on the support of invariant measures for dynamical systems with semialgebraic dynamics and constraints. Invariant measures have been studied extensively in dynamical systems theory [12] and Markov decision processes [10] and it is now recognized that key properties of a dynamical system can be assessed by considering only a few moments of a measure transported along the system flow [1]. The constructive proof of the ergodic partition theorem [3][19] provides characterization of ergodic sets, which are the smallest invariant sets that ensure measurability of partition. Ergodic sets are the supports of ergodic measures, that can in turn be studied via their moments, or their Fourier coefficients in the
periodic case. Here too, the key idea consists in observing the action of invariant measures on a countable number of observables, or test functions, see e.g. [5] or [13]. Even more recently, invariant measures and weak Kolmogorov–Arnold–Moser (KAM) theory have been used to study geometrical properties of the joint spectral radius (JSR) of a set of linear operators [6, 21].

Hierarchies of finite-dimensional convex optimization problems have been introduced in the early 2000s to solve numerically non-convex optimization problems with semialgebraic data, with convergence guarantees [13]. The overall strategy consists of building a family of semidefinite programming (SDP) problems [2] of increasing size, with primal SDP problems relaxing the original polynomial optimization problem in the space of truncated moments of a measure supported on the globally optimal solutions, and dual SDP problems certifying bounds on the global optimum with specific polynomial sum-of-squares (SOS) representations. In the context of polynomial optimization, this is called the moment-SOS hierarchy [15] or sometimes Lasserre’s hierarchy [22], and this relies on fundamental results of convex algebraic geometry, see [24] or [3]. The approach has been extended in [14] to optimal control problems on ordinary differential equations (ODEs), and more recently, to construct families of semialgebraic approximations of the support of measures transported along the flow of controlled ODEs [9].

Another topic of interest was the relationship between geometric properties of dynamical systems and spectral properties of the associated operators. In fact, the hallmark of the work on the operator-theoretic approach in the last two decades is the linkage between geometrical properties of dynamical systems - whose study has been advocated and strongly developed by Poincaré and followers - with the geometrical properties of the level sets of Koopman eigenfunctions [16,17,19]. The operator-theoretic approach has been shown capable of detecting objects of key importance in geometric study, such as invariant sets, but doing so globally, as opposed to locally as in the geometric approach. It also provides an opportunity for study of high-dimensional evolution equations in terms of dynamical systems concepts [20,25] via a spectral decomposition, and links with associated numerical methods for such evolution equations [26].

Judging by the contents of the lively discussions during lectures or daily ad-hoc seminars, sometimes extended to the late hours of evening, we believe that the workshop was a success. It has offered a timely and unique opportunity of collaboration and exchange of views among experts in operator theory, convex optimization, dynamical systems, and systems control. The seeds of a new research group, strongly bonded by convergent mathematical interests, were laid on this occasion.

The abstracts below offer an accurate picture of the scientific themes touched during the mini-workshop.

Didier Henrion
Igor Mezić
Mihai Putinar
Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

References

Mini-Workshop: Applied Koopmanism

Table of Contents

Amir Ali Ahmadi
Polynomial optimization for analysis of dynamical systems ............... 305

Marko Budíšić (joint with Mihai Putinar)
Regularizing singular measures for maximum entropy reconstruction ... 308

Stéphane Gaubert (joint with X. Allamigeon, P. Benchimol and M. Joswig)
Nonarchimedean linear programming and mean payoff games ............... 309

Didier Henrion
The Lasserre hierarchy for polynomial optimization and optimal control 311

Roxana Hess (joint with Didier Henrion, Jean B. Lasserre, Tien Son Pham)
Semidefinite approximations of the polynomial abscissa .................. 312

Oliver Junge
Set oriented numerics for dynamical systems .................................. 313

Raphaël Jungers
Joint spectral characteristics of matrix semigroups ......................... 315

Milan Korda (joint with D. Henrion, Colin N. Jones)
Moment-sum-of-squares hierarchies for set approximation and optimal control ............................................................... 317

Kari Küster
Pure Koopmanism ................................................................. 320

Uwe Küster
Approximation of the Koopman-Operator eigenvalues of Trajectories ... 322

Yuri Latushkin
An invitation to evolution semigroups and linearized Euler operator ... 325

Alexandre Mauroy (joint with Igor Mezić)
Analysis of dissipative nonlinear systems using the eigenfunctions of the Koopman operator ......................................................... 327

Igor Mezić
The Koopman Operator Formalism ............................................ 328

Ryan Mohr (joint with Igor Mezić)
Dissipative Dynamics, Spaces of Observables for the Associated Koopman Operator, and the GLA Theorem ................................. 331

Mihai Putinar (joint with Marko Budíšić)
Moment problems, maximal entropy and exponential transforms .......... 334
Clarence W. Rowley (joint with Matthew O. Williams and Ioannis G. Kevrekidis)

*Connections between Koopman and Dynamic Mode Decomposition* . . . . 337