Mixed-integer Nonlinear Optimization: A Hatchery for Modern Mathematics

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Abstract. The aim of this workshop was fostering the growth of new mathematical ideas arising from mixed-integer nonlinear optimization. In this regard, the workshop has been a resounding success. It has covered a very diverse scientific landscape, including automated proof in computational geometry, the analysis of computational complexity of MINO in fixed and variable dimension, the solution of infinite MINO such as those appearing in mixed-integer optimal control, the theoretical and computational deployment of traditional integer and continuous approaches to achieve new solution algorithms for large-scale MINO, a classification of the most interesting engineering and technology applications of MINO, and more. It has synthesized twenty open questions and challenges which will serve as a roadmap for the years to come.

Mathematics Subject Classification (2010): 90C06, 90C11, 90C22, 90C26, 90C30.

Introduction by the Organisers

Mixed-Integer Nonlinear Optimization (MINO) is a subfield of Mathematical Optimization (MO) which studies formulations involving both integer and continuous variables, as well as linear and nonlinear functions on these variables, and the methods used to find their solutions.

Definition. MINO can be seen as a formal language used to describe a very large class of optimization problems. Each valid sentence of this language is called a formulation. Formulations consist of decision variables, parameters, objectives
and constraints, and are written as follows:

\[
\begin{align*}
\min & \quad f(x, p) \\
\forall i \in I & \quad g_i(x, p) \leq 0 \\
\forall j \in Z & \quad x_j \in \mathbb{Z},
\end{align*}
\]

where \( I, Z \) are index sets. The parameter vector \( p \) lists elements from a number field (e.g. \( \mathbb{Q} \)), and encodes the input of the problem (also called instance). The decision variable vector \( x \) is a list of variable symbols; it encodes the output of the problem (also called the solution). The function symbols \( f, g_i \) (for each \( i \in I \)) are valid sentences of another formal language having \( x_1, x_2, \ldots \) and elements of \( \mathbb{Q} \) as atoms, which can be recursively combined using arithmetic operators (including powers) and a few transcendental functions such as log, exp and so on. The last constraint in Eq. (1) expresses an integrality requirement on the variables indexed in \( Z \) (the other variables are assumed to be continuous).

**Motivation.** The interest of expressing optimization problems formally by means of MO formulations such as Eq. (1) is that there are solution methods which address all instances in a certain class. Specifically, one can solve Eq. (1), at least in practice, using a range of rather powerful solver algorithms, such as spatial Branch-and-Bound. This shifts the focus from designing and implementing algorithms for solving problems (which is hard) to modelling an application using the formal language (which is easier). All sorts of problems arising in industry, science, technology can be modelled as MINO, but there are many possible MINO representations of a given optimization problem, and not all of them yield the same solver performance. A crucial problem is then that of reformulation, which aims at finding MINO representations which are good from the point of view of the solver (see e.g. J.P. Vielma’s talk). In particular, humans model using quantifiers over index sets in order to express properties of indexed variables and indexed parameters, whereas solvers require unquantified input. Quantified formulations are also called structured, whereas the solver input format is known as flat.

**Organization.** The workshop consisted of five tutorial-type talks (45’ followed by 15’ discussion), twenty invited talks (30’ followed by 15’ discussion), and eighteen short research announcements (10’ followed by 5’ discussion). We also organized two optional sessions: one on open problems and challenges, and a second one about the mathematically-oriented computer language Julia, with its mathematical programming extension JuMP. Among the most interesting open problems, we emphasize a stress on extended formulation size and complexity, verification of copositivity and complete positivity, the solution of a variety of small, but very hard, mostly geometrical MINO problems, automatically recognizing some structural features of a given formulation, finding tight convex relaxations of nonconvex functions that are hard to optimize, dealing with black-box nondifferentiable functions.

Each day started with one of the tutorial talks, then continued with a variety of invited talks focusing mainly on the pillars of our hatchery for modern mathematics: hierarchies of approximations, mixed-integer nonlinear optimal control,
the power of lifting, big data. On Tuesday we scheduled an afternoon session with six research announcements, and on Thursday we had nine. Wednesday afternoon saw the traditional hike, and Friday was a full working day.

**Topics of the tutorial talks.** Mixed-integer control problems, such as those arising in the control of chemical plants or automatic vehicles, are among the most difficult in the MINO arena, due to their potentially large size, the range of nonlinearities which appear in the problem function forms, and the fact that the constraints are often differential equations and/or nondifferentiable “black-box” functions (see e.g. Armin Fügenschuh’s and the Simons fellow Sven Leyffer’s talks).

Currently, one of the topics which draws the most attention is solving polynomial MINO problems, ranging from quadratic (see e.g. A. Del Pia’s talk) to general polynomial. In the latter case, Semidefinite Programming (SDP) formulations are often employed — these are MO formulations involving positive semidefiniteness of a certain matrix involving some decision variable symbols (see A.A. Almadi’s talk). More precisely, the most promising approach to provide valid bounds are relaxation hierarchies, such as Lasserre’s (who was part of the audience at the workshop — also see E. De Klerk’s talk).

Interestingly for mathematics, MINO type problems can also be used to derive proofs. Specifically, many geometrical problems can be cast as MINO; there exist various techniques for turning such solutions into proofs, as shown by some of the speakers at this workshop (e.g. F. Vallentin’s talk).

Finally, the current trend emphasizing the availability of increasingly large amounts of data from security, retail, social networking and other sources suggests the possibility of finding relationships between many data sources, and exploit them in a concerted or integrated way. This presents the enormous difficulty of having to not only solve enormous optimization problems, but also that of leveraging the data to actually yield, or at least validate, the formulation (see A. Lodi’s talk).

**The short research announcements.** We cover SRAs here since they are not included in the abstracts below.

J. Linderoth presented GūBoLi, a new solver for nonconvex box-constrained quadratic programs using Integer Programming (IP) software technology. L. Hupp discussed IP approaches for structured binary quadratic optimization problems, with special attention to quadratic matching. S. Sorgatz presented results on improving the flow of vehicular traffic at traffic light intersections. R. Misener presented a technique for automatically recognizing pooling problem structure in arbitrary flat MINO problems. S. Onn talked about lexicographics combinatorial optimization. M. Firsching, who was not part of our workshop but was working on the editorial board of MFO’s snapshot program, presented a technique for turning floating point solutions of MINO problems into minimal polynomials of the correct algebraic number. F. Liers provided a structural investigation of piecewise linearized network flow problems. K. Anstreicher proved that the SDP relaxation of quadratic optimization with ellipsoidal hollows is exact if the SDP
relaxation of the same problem without hollows is exact. S. Wiese studied indicator constraints in the linear job-shop scheduling problem. L. Mencarelli presented a multiplicative weights update algorithm for MINO problems. Ky Vu introduced a new randomized algorithm for restricted linear membership problems based on random projections. A. Gupte discussed explicit disjunctive inequalities for some structured nonconvex sets. S. Weltge presented new formulations and valid inequalities for the AC optimal power flow problem. S. Weltge discussed the size of SDP extensions. A. Martin’s talk was about an ongoing effort within a large grant for network problems including physical transport. P. Belotti discussed the impact of the presolver in solution algorithms for MINO problems. Finally, P. Bonami gave a talk about solving empty mixed integer second-order conic programs (MISOCP) using the CPLEX solver.

**A need.** One of the most senior members of the MINO community, I. Grossmann, expressed a need for developing a conceptual roadmap showing the interconnections of all the theoretical subproblems that are being addressed by the various MINO researchers. This should ultimately lead to better solutions of general MINO and mixed-integer nonlinear control problems. This accomplishment would provide the MINO field with a stronger theoretical foundation.

**The future.** In view of the very stimulating interaction between the researchers during the workshop and of the presumably ongoing strong interest in symmetries in optimization problems (as demonstrated by the many open directions of future research), the workshop participants strongly agreed that a similar meeting in two years would be most desirable. We are exploring various possibilities with other mathematical centers such as CIRM in Marseille, Cargèse in Corsica, Bertinoro in Italy, and so on. The idea of submitting another application to the MFO in three or four years was received enthusiastically, and encouraged by the Director of the MFO.

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