Abstract. The dynamical (or von Neumann) spectrum of a dynamical system and the diffraction spectrum of the corresponding measure dynamical system are intimately related. While their equivalence in the case of pure point spectra is well understood, this workshop aimed at an appropriate extension to systems with mixed spectra, building on recent developments for systems of finite local complexity and for certain random systems from the theory of point processes. Another focus was the question for connections between Schrödinger and dynamical spectra.

Mathematics Subject Classification (2010): 37B05, 52C23, 81Q10.

Introduction by the Organisers

The spectral theory of dynamical systems was initiated by Koopman [19] and von Neumann [24], and later developed in various directions. So far, a complete classification result in the realm of measure-theoretic dynamics only exists for dynamical systems with pure point spectrum. This approach is usually formulated via the Halmos–von Neumann theorem [17]. This certainly applies to model sets (also known as cut and project sets), where the corresponding Kronecker factor emerges constructively [8, 4].

The development of (mathematical) diffraction theory [18] admits an alternative approach to measure dynamical systems on locally compact Abelian groups. For systems with pure point spectrum, the equivalence of the two approaches has been established in a series of publications [22, 20, 7, 21]. More recent are first steps towards an extension to systems with mixed spectra [9], with particularly concrete
results for systems with finite local complexity (FLC) [2, 5, 10] (compare also the abstracts by D. Lenz, M. Baake, U. Grimm and F. Gähler below).

The essential point here is the insight that, in general, the dynamical spectrum is richer than the diffraction spectrum of a system [23, 10], but not than the collection of diffraction spectra of the system and a suitable family of its factors [9]. In general, one has to expect that such a family is infinite, at least if one restricts to factors of the same complexity type. Astonishingly, in many of the classic examples, one can work with a finite family; see [4, 3] and references therein, as well as the abstract by D. Lenz below. What is presently lacking is a classification of those systems where such a finiteness condition applies. Even a useful sufficient criterion is unknown at present.

The primary aim of this mini-workshop was to bring together experts from both ends of the spectrum in order to reach a better understanding of the correct equivalence notion and to take first steps towards a spectral classification beyond the pure point case. To facilitate discussions, the mini-workshop started with four survey talks which set the scene on central topics such as almost periodic measures, dynamical and diffraction spectra, Schrödinger spectra and statistical mechanics approaches; see the abstracts by N. Strungaru, D. Lenz, D. Damanik and A.C.D. van Enter for details. A further three talks addressed various topological aspects of tilings and tiling spaces; compare the abstracts by J. Kellendonk, T. Fernique and A. Clark. The talks of the remaining ten participants discussed specific questions related to spectral properties, as detailed elsewhere in this introduction.

Particularly interesting questions in this context concern systems with singular continuous spectra, which have been studied in the context of Schrödinger operators for some time (see, e.g., [12] and references therein, and compare the abstracts by D. Damanik, A. Gorodetski, W. Yessen and J. Fillman), as well as systems with absolutely continuous spectra, as they appear in the theory of point processes [16, 1]. In the latter case, it quite often occurs that dynamical and diffraction spectra become equivalent again [6]. In fact, this and various heuristic arguments (compare the abstracts by A.C.D. van Enter and H. Kösters) point towards the conjecture that the diffraction spectrum is absolutely continuous relative to the maximal spectral measure of the dynamical spectrum.

A more recent extension of the theory emerged by the use of exact renormalisation techniques for the spectra (compare the abstracts by M. Baake and F. Gähler), which can help significantly to determine the spectral type, and by the study of weak model sets (see the abstracts by C. Richard and C. Huck), which fail to be minimal as dynamical systems. Moreover, they have entropy, but can still show pure point spectrum. This is, in a way, in contrast to highly ordered systems such as the Rudin–Shapiro chain and its generalisations via Hadamard matrices (compare the abstract by N. Frank), which show Lebesgue measures in their dynamical and diffraction spectra.

Another focus of the mini-workshop was to take first steps to clarify the connection between Schrödinger spectra and the dynamical and diffraction spectra discussed above. There is very little understanding of this question in general,
aside from some heuristics based on the existing results on the two sides. Namely, based on the known results for some specific classes of examples, it is quite apparent that with increasing disorder of the system, the dynamical and diffraction spectra become more regular (i.e., more continuous), while the Schrödinger spectra become more singular. Finding a direct connection between the two sides, perhaps a suitable duality notion, explaining these tendencies would be highly desirable. Thus, a secondary aim of the proposed mini-workshop was to facilitate and stimulate discussions between those participants working on dynamical and diffraction spectra and those (also) working on Schrödinger operators.

One recent advance in this direction has been obtained in [13, 14], where (for the central model in the context of quasicrystals, the Fibonacci case) the Schrödinger density of states measure, which is the phase average of the spectral measures associated with the Fibonacci Schrödinger operator, was shown to result from the measure of maximal entropy of the associated trace map dynamical system by projection along the stable manifolds of points in the non-wandering set of the trace map. While this result does establish an explicit connection between spectral measures and dynamical measures, it is somewhat special to the Fibonacci case as it makes use of the presence of a hyperbolic basic set for the map implementing the self-similarity. Such a structure is not present in more general settings.

As a result of the discussions that took place during the mini-workshop, several research projects are now under way. One goal is to find a closer connection between those systems that are dynamically pure point and the corresponding Schrödinger spectra. For the most prominent examples, such as the Fibonacci or the period doubling case, it is known that the singular continuous nature of the Schrödinger spectral measure is uniform across the hull [11, 15]. In contrast, no such uniform result is known for cases whose dynamical spectrum is not pure point. It is therefore natural to explore whether there is a connection between these phenomena, and hence to look for further models with pure point dynamical spectrum and uniform singular continuous Schrödinger spectrum, as well as a better understanding of whether models with uniform singular continuous Schrödinger spectrum that do not have pure point dynamical spectrum can or cannot exist.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

References

Mini-Workshop: Dynamical versus Diffraction Spectra in the Theory of Quasicrystals

Table of Contents

Nicolae Strungaru (joint with Robert V. Moody)
   Almost periodic measures and Meyer sets .......................... 2993

Daniel Lenz (joint with Michel Baake, Aernout C.D. van Enter)
   Dynamical versus diffraction spectra in aperiodic order ........ 2994

David Damanik
   Schrödinger spectrum and quantum dynamics ......................... 2995

Aernout C.D. van Enter
   Aperiodicity in statistical mechanics ............................. 2996

Michael Baake (joint with Natalie Frank, Franz Gähler, Uwe Grimm, E. Arthur Robinson)
   Autocorrelation via renormalisation ................................ 2997

Johannes Kellendonk (joint with Lorenzo Sadun)
   Conjugacies of FLC Delone sets .................................... 2998

Thomas Fernique
   From random to aperiodic tilings .................................... 2999

Anton Gorodetski
   Sums of Cantor sets and the square Fibonacci Hamiltonian ........ 3000

William Yessen (joint with Jake Fillman, May Mei, Yuki Takahashi)
   Tridiagonal substitution Hamiltonians ................................ 3001

Jake Fillman (joint with David Damanik, Anton Gorodetski)
   Continuum models of one-dimensional quasicrystals .............. 3002

Uwe Grimm (joint with Michael Baake)
   Binary bijective block substitutions in d dimensions ............. 3004

Franz Gähler
   A decorated silver mean tiling with mixed spectrum ............... 3005

Christian Huck (joint with Michael Baake)
   Ergodic properties of visible lattice points ....................... 3006

Christoph Richard (joint with Christian Huck)
   On pattern entropy of model sets ................................... 3007

Natalie Priebe Frank
   Substitution $\mathbb{Z}^d$ sequences with non-simple Lebesgue dynamical spectrum 3008
Holger Kösters (joint with Michael Baake and Robert V. Moody)

Diffraction spectra and dynamical spectra of some random point sets . . . 3009

Alex Clark

Topological perspective on the dynamics of tilings ......................... 3010