Abstract. Metrics of special holonomy are of central interest in both Riemannian and complex algebraic geometry. We focus on an important classification problem of a particular type of special holonomy manifolds, namely compact quaternion-Kähler with positive scalar curvature (Salamon-LeBrun conjecture). In the language of algebraic geometry this corresponds to the classification of Fano contact manifolds. By bringing together leading experts in both fields this workshop pursued a two-fold goal: First, to revise old and to develop new strategies for proving the most central conjecture in the field of quaternionic Kähler geometry. Second, to introduce young researchers at PhD/PostDoc level to this interdisciplinary circle of ideas.


Introduction by the Organisers

This mini-workshop was organised by Anna Fino (Università di Torino), Uwe Semmelmann (Universität Stuttgart), Jaroslaw Wiśniewski (Uniwersytet Warszawski) and Frederik Witt (Universität Münster). We had 16 participants (four of which at junior level) to discuss new and old approaches to the Salamon-LeBrun conjecture. Its Riemannian side asserts that any compact quaternion-Kähler with positive scalar curvature is necessarily symmetric. On the algebraic side this matches the conjecture that any Fano contact manifold is homogeneous.

To explain this link briefly, let us recall that metrics of special holonomy, such as quaternion-Kähler metrics, are of central interest in both Riemannian and complex algebraic geometry. This is evident in the case of Kähler and Calabi-Yau
metrics where the metric induces a distinguished complex structure. The case of hyperkähler and quaternionic-Kähler manifolds is less clear. For hyperkähler metrics we have an \( S^2 \) worth family of complex structures which in general are not biholomorphic to each other, while for quaternionic-Kähler metrics there is in general no complex structure at all. However, a generalisation of the Atiyah-Hitchin-Singer twistor space construction in dimension four to higher dimensions associates with these metrics a well-defined complex manifold. For instance, if \((M^{4k}, g)\) is a quaternionic-Kähler manifold of real dimension \(4k\), then its associated twistor space \(Z\) is a complex contact manifold of complex dimension \(2k+1\). Moreover, \(Z\) carries a positive Kähler-Einstein metric if \(M\) is compact with positive scalar curvature. In particular, \(Z\) is Fano.

Existence of special holonomy metrics, in particular non-symmetric ones, is one of the central issues of the theory. For instance, Berger’s original classification of non-symmetric metrics included the case of \(\text{Spin}(9)\) which subsequently could be ruled out. Similarly, the only known examples of compact quaternion-Kähler manifolds with positive scalar curvature are symmetric. The Salamon-LeBrun conjecture (which for instance is true in dimension 8) asserts that these are the only examples. Translated into complex geometry via the twistor construction this boils down to show that any Fano contact manifold is homogeneous.

On the other hand, contact structures on complex projective manifolds are very rare. By results of Demailly, Kebekus, Peternell, Sommese and Wiśniewski, if a contact projective manifold admits a contact structure and its second Betti number is \(> 1\) then the manifold in question is the projectivisation of the (co)tangent bundle over another projective manifold and the contact structure is the natural one. Thus, such manifolds seem to be exceptional and it is plausible to expect that the only known examples constitute the complete list of such manifolds.

The workshop started off with a couple of introductory lectures by Witt and Simon Salamon (who joined us via a video conference) on quaternionic-Kähler manifolds and the twistor construction, and by Wiśniewski on Fano contact manifolds. We then had more specialised lectures dealing with specific issues.

On the differential geometric side, Semmelmann and Weingart talked about representation theoretic methods for proving Weitzenböck formulae and vanishing theorems. This linked also into Dessai’s talk on quaternion-Kähler manifolds in dimension 12 where an application of the index theoretic and topological ideas successful in dimension 8 have failed so far. The talks by Swann, Cortés and Bielawski dealt with the construction of quaternion-Kähler metrics. Finally, Amann and Moroianu talked about important properties of quaternion-Kähler manifolds, namely formality and the non-existence of almost complex structures.

On the algebraic side, apart of the Wiśniewski’s lecture which provided a general introduction to contact Fano manifolds, there were four lectures about specific algebraic geometry methods for studying such manifolds. Buczyński, Hwang and Kebekus lectures were about rational curves on contact Fano manifolds. Kebekus used families of minimal rational curves to recover the contact distribution. Hwang explained his results and expectations regarding the rôle of the variety of minimal
rational tangents (or VMRT) of a contact manifold in the classification of such manifolds. On the other hand, Buczyński’s talk was about the singularities of minimal rational curves on contact manifolds. The lecture by Campana concerned a different approach to the classification problem. Namely, Campana presented results by Clemens and Ran concerning generic semipositivity of sheaves of differential operators on Fano manifolds.

During the week we had intensive discussions on old and new approaches to this conjecture with stimulating exchanges between Riemannian and algebraic geometers during and after the talks. We also noted with pleasure the high commitment and very active presence of the young participants. As a result we are planning a sequel to this workshop in the near future. Finally we are happy to acknowledge the never failing support and professional handling by the entire Oberwolfach stuff.
Mini-Workshop: Quaternion Kähler Structures in Riemannian and Algebraic Geometry

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