Mini-Workshop: Spherical Varieties and Automorphic Representations

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Abstract. This workshop brought together, for the first time, experts on spherical varieties and experts on automorphic forms, in order to discuss subjects of common interest between the two fields. Spherical varieties have a very rich and deep structure, which leads one to attach certain root systems and, eventually, a “Langlands dual” group to them. This turns out to be important for automorphic forms, as it provides a (mostly conjectural) way to analyze periods of automorphic forms and related problems in local harmonic analysis.

Mathematics Subject Classification (2010): 22 (primary), 11, 14 (secondary).

Introduction by the Organisers

The goal of this workshop was to bring together, for the first time, experts on two different fields: spherical varieties, on one hand, and periods of automorphic forms, on the other.

If $G$ is a reductive group over a global field $k$ (e.g.: a number field), automorphic forms are (roughly speaking) elements of irreducible representations for the action of $G(\mathbb{A}_k)$ on $C^\infty([G])$, where $\mathbb{A}_k$ denotes the ring of adeles of $k$, and $[G] = G(k)\backslash G(\mathbb{A}_k)$. (The reader may think of $G(k)\backslash G(\mathbb{A}_k)$ as a homogeneous manifold such as $\text{GL}_n(\mathbb{Z})\backslash \text{GL}_n(\mathbb{R})$, whose functions are equipped with more symmetries than the action of $G(\mathbb{R})$, the so-called Hecke operators.)

Automorphic representations have important invariants which are very difficult to study, such as their $L$-functions. A common way to study automorphic $L$-functions is to take large enough – spherical – subgroups $H \subset G$ and to consider
period integrals of automorphic forms, i.e. integrals of the form:

\[ \int_{[H]} \phi(h) dh, \]

perhaps adding a continuously varying character of \([H]\). Such integrals are often equal to \(L\)-functions or special values of \(L\)-functions, and they also reveal other interesting properties of the automorphic representation \(\pi\) of \(\phi\), such as being a “functorial lift” in the sense of Langlands.

It turns out that these phenomena are related to the structure of spherical varieties discovered in works of Luna, Vust, Brion, Knop, and others, and to the dual group \(\mathcal{G}_X\) that was associated by Gaitsgory and Nadler to any spherical variety \(X\), based on this structure.

The goal of our workshop was to inform experts on spherical varieties about the general theory and relevant problems in automorphic forms, and vice versa. Automorphic representations appear in harmonic analysis of the homogeneous space \([G]\), and their local constituents appear in harmonic analysis on the symmetric space \(G(k_v)\), where \(v\) is any completion of \(k\); Wee Teck Gan gave a general introduction to non-abelian harmonic analysis, with an emphasis on the Plancherel decomposition of \(L^2(G(k_v))\). This introduction was continued by Dipendra Prasad, who talked about automorphic representations, the local Langlands Conjecture, and automorphic \(L\)-functions. On the side of spherical varieties, Guido Pezzini gave an introduction to their structure and invariants, and Paolo Bravi described the “Luna systems”, i.e. these combinatorial invariants which are used to classify wonderful and spherical varieties. The relation between harmonic analysis and the compactification theory started appearing in the talk of Yiannis Sakellaridis, who explained the role of “boundary degenerations”, i.e. normal bundles to \(G\)-orbits in suitable compactifications, to the description of the continuous spectrum in the Plancherel formula for \(L^2(X(k_v))\). There are several root systems that one can attach to a spherical variety, always with the same Weyl group and on the same vector space but with roots of different length, encoding different features of the variety; Bart Van Steirteghem compared the various root systems and explained their use. Proving that these invariants give rise to root systems, though, is quite involved, and Friedrich Knop recounted his analysis of the moment map (the graded version of his analysis of invariant differential operators), which provides a conceptual proof for the appearance of root systems. Bernhard Krötz described problems in real harmonic analysis related to spherical varieties and “real spherical varieties”, a term he uses for homogeneous spaces for a real Lie group on which a minimal \(\mathbb{R}\)-rational parabolic (not necessarily a Borel) acts with an open orbit. One of the ways to understand the root system of an affine spherical variety is as a measure of the failure of the coordinate ring – which is naturally filtered by dominant weights – to be graded; on the third day, Michel Brion described the invariant Hilbert scheme, which roughly describes the possible ring structures on a given \(G\)-module. On the automorphic side, Omer Offen introduced the relative trace formula of Jacquet, a basic tool for studying period integrals, but possibly also the natural setting in which conjectures about spherical varieties and their dual
group should be formulated. On the fourth day, Yiannis Sakellaridis explained the Satake isomorphism, the work of Gaitsgory and Nadler, and the relevance of finer invariants, such as colors, in the study of unramified functions on spherical varieties. There is a very simple and telling way in which the Weyl group of a spherical variety appears, and this is by considering an action, defined by Friedrich Knop, of the full Weyl group on the set of Borel orbits; Jacopo Gandini described this action, as well as further results on the structure of Borel orbits. On the last day Farrell Brumley explained the number-theoretic importance of periods via some examples from analytic number theory, and Nicolas Templier described results on the asymptotic behavior of Whittaker functions. Finally, Stephanie Cupit-Foutou has used the invariant Hilbert scheme to classify spherical varieties, and she gave an alternative description of Luna data based on the invariant Hilbert scheme.
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