Abstract. The workshop *Mathematical Billiards*, organised by Sergei Tabachnikov (Penn State) and Serge Troubetzkoy (Marseille) was held April 4th–April 10th, 2010. This meeting was well attended by over 40 participants including a number of master and PhD students, with broad geographic representation. This workshop was a nice blend of researchers with various backgrounds who brought in their various point of views to cover the classics as well as recent advances in mathematical billiards and flat surfaces.

The report consists in the abstracts for the 18 lectures, followed by the abstracts for the 4 short talks that took place in the evenings. During the workshop, there was also a demo of the mathematical software Sage.

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Introduction by the Organisers

**The Billiard system.** The billiard dynamical system describes the motion of a free particle in a domain with a perfectly reflecting boundary.

More technically, a billiard table $Q$ is a subset of a Riemannian manifold (usually $\mathbb{R}^2$) with a piece-wise smooth boundary. We define the billiard flow as follows: the billiard ball is a point particle, it moves along geodesic lines in $Q$ with elastic collisions with $\partial Q$. The latter means that, at the impact point, the velocity vector of the particle is decomposed into two components, tangential and normal to $\partial Q$; then the normal component instantaneously changes signs, whereas the tangential component remains the same, after which the free motion continues. In dimension two, this is the famous law of geometrical optics: the angle of incidence equals the angle of reflection.
Many mechanical systems with elastic collisions, that is, collisions preserving the total momentum and energy of the system, reduce to billiards. Perhaps the most famous example is an idealized gas made of massive elastically colliding balls. Here is an interesting lesser known example: the system of three elastically colliding point masses on a circle reduces, after fixing the center of mass, to the billiard inside an acute triangle whose angles depend on the ratios of masses. There are many physically motivated variations on billiards, such as magnetic billiards, in which free particles are subject to the action of a magnetic field.

The dynamical behavior of billiards is strongly influenced by the shape of the boundary. Billiards naturally fall into three classes: depending on whether the pieces of the boundary curve out, curve in, or are flat. In each of the cases the mathematical machinery used in the study is quite different. The presentation of talks below is organized accordingly.

The final group of talks will study outer (also known as dual) billiards, which are played outside a convex table $Q$ in the Euclidean plane. Dual billiards are defined as follows. Fix an orientation of $Q$. Given a point $x$ outside $Q$, draw the segment $xy$, with $y \in Q$, of the tangent line to $Q$ such that its orientation agrees with that of $Q$. Extend this segment through $y$ to the point $T(x)$ such that $\text{dist}(Tx, y) = \text{dist}(x, y)$. The map $T$ of the exterior of $Q$ to itself is the dual billiard transformation. This map is area-preserving; its definition extends to the spherical and hyperbolic geometry (in the former, outer and inner billiards are equivalent via the spherical duality). Outer billiards can be also defined in even-dimensional Euclidean spaces.

The following books are devoted to billiards: [1, 3, 8, 9].

**Hyperbolic billiards.** If the boundary of $Q$ curves out, then parallel incoming orbits scatter, or disperse, producing hyperbolic behavior. A second mechanism of hyperbolicity exists: if two smooth curving in components are placed sufficiently far apart, then parallel orbits first focus, but then have time to diverge before the next collision. One of the main motivations of the study of hyperbolic billiards is Boltzmann’s ergodic hypothesis, see [7].

The mathematical tools used to study hyperbolic billiards are the same as the ones used to study hyperbolic dynamical systems (Anosov systems, Axiom A systems, expanding maps, etc). There are serious additional difficulties, the presence of singularities (tangent orbits and orbits hitting non-smooth points of the boundary).

**Elliptic billiards.** The billiard in an ellipse is completely integrable: a subset of full measure in its phase space is foliated by invariant curves, corresponding to the billiard trajectories tangent to confocal ellipses and hyperbolae, the caustics of the billiard system (note however that one leaf of this foliation is singular: this is the invariant curve consisting of the trajectories that pass through the foci of the ellipse). Similar complete integrability holds for billiards inside ellipsoids in multi-dimensional space.

It turns out that part of this structure is shared by arbitrary convex tables. Lazutkin showed that one can apply the celebrated KAM theorem to show that
a set of positive measure of caustics exist for sufficiently smooth tables. Birkhoff showed that periodic orbits always exists in plane billiards with sufficiently smooth boundary with positive curvature. On the other hand, Mather proved that if the curvature vanishes at a point then the billiard possesses no caustics.

**Polygonal billiards.** Billiards in polygons come in two classes: rational and irrational polygons. Rational polygons are those for which the angles between sides are rational multiples of $\pi$. A rational billiard table determines a flat surface, this construction allows one to use the tools of Teichmüller theory to study rational billiards, and many deep results have been obtained this way, see, e.g., [4, 10]. Most of the polygonal talks will be on rational polygons, since in the irrational case there is essentially no machinery available, other than elementary geometry and computer simulation. As a consequence, the available results are considerably more scarce.

**Dual or outer billiards.** In the first volume of the Mathematical Intelligencer, Jurgen Moser wrote an article proposing the outer billiard as a toy model to study the question of the stability or not of the solar system [5]. The recent progress in the study of polygonal outer billiards is the subject of the two talks. See [2] for a survey of outer billiards and [6] for a monograph devoted to a special class of quadrilaterals, the irrational kites, for which some outer billiard orbits escape to infinity.

**References**


