Abstract. This is a short report on the conference "Teichmüller Space (Classical and Quantum)" held in Oberwolfach from May 28th to June 3rd, 2006.

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Introduction by the Organisers

In a broad sense, the subject of Teichmüller theory is the study of moduli for geometric structures on surfaces. The progenitor of the subject is usually considered to be G. F. B. Riemann, who in a famous paper on Abelian functions, studied the moduli space of algebraic curves and stated that the space of deformations of equivalence classes of conformal structures on a closed orientable surface of genus $g \geq 2$ is of complex dimension $3g - 3$. This was explicated by O. Teichmüller who laid the foundations of the theory in a series of famous papers (during a remarkably brief period). Many prominent mathematicians including L. Ahlfors and L. Bers continued developing the theory over several decades. In the 1970s, W. Thurston introduced techniques of hyperbolic geometry in the study of Teichmüller space and its asymptotic geometry. In the 1980s, new combinatorial treatments of Teichmüller and moduli spaces evolved with a concurrent interplay of ideas from string theory in high-energy physics.

Teichmüller theory is one of those precious subjects in mathematics which have the advantage of bringing together, at an equally important level, fundamental ideas coming from different fields. Among the fields associated to Teichmüller theory, one can surely mention complex analysis, hyperbolic geometry, discrete
group theory, algebraic geometry, low-dimensional topology, Lie groups, symplectic geometry, dynamical systems, topological quantum field theory, string theory, and many others. Teichmüller theory is growing at a fantastic rate, and the fact that it involves all these areas is probably a consequence of the fact that Teichmüller space itself carries a diversity of rich structures. As a matter of fact, this space can be seen from at least three points of view: as a space of equivalence classes of hyperbolic metrics, as a space of equivalence classes of conformal structures, and as a space of equivalence classes of representations of the fundamental group of a surface into a Lie group. Each of these points of view endows Teichmüller space with various structures, including several interesting metrics, a natural complex structure, a symplectic structure, a real analytic structure, an algebraic structure, cellular structures, various boundary structures, a natural discrete action by the mapping class group, interesting geodesic and horocyclic flows on the quotient Riemann moduli space, a quantization theory of its Poisson structure, and the list goes on and on.

The quantization of Teichmüller space was developed in the last few years by L. Chekhov and V. Fock and independently in work by R. Kashaev. This theory produces noncommutative families of deformations of the Poisson or symplectic structure of Teichmüller space in the form of $\ast$-algebras, with an action of the mapping class group of the surface as an outer automorphism group. In particular, quantization of Teichmüller space leads to new invariants of hyperbolic three-manifolds.

The conference brought together people in almost all of the active areas of Teichmüller theory. The fact that Teichmüller theory is a living and rich subject connecting several areas of mathematics was reflected in the richness of the talks that were presented, and in the variety of the new perspectives that were discussed at the problem session, on which we report separately.

We note that many other attendees were ready to give interesting talks than time permitted. As a general rule, younger researchers were given the opportunity to present their own work. In this short report, we have divided the talks that were delivered in five groups:

1) Metric theory. U. Hametsädt reported on her recent work on the behaviour in moduli space of images of certain closed geodesics for the Teichmüller metric, namely, for every compact set $K$, one can find such images which do not intersect $K$. G. Schmithüsen reported on Teichmüller disks, which are embeddings of the Poincaré disks which are isometric with respect to the Teichmüller metric. G. Théret gave a talk on Thurston’s asymmetric metric on Teichmüller space and presented results on the convergence of certain geodesics to points on Thurston’s boundary.

2) Mapping class groups and the associated simplicial complexes. V. Markovich gave a review of several realization problems for the mapping class group and he reported on his result stating that for any closed surface $S$ of genus $\geq 6$, the natural projection from the space of homeomorphisms to the mapping class group has no section. This result answered a famous open problem. D. Kotschick gave
a survey of his work on quasi-homomorphisms with applications to the mapping class group. J. McCarthy gave a talk in which he described the automorphism group of a recently introduced simplicial complex, the complex of domains of a surface (joint work with A. Papadopoulos). E. Irmak described recent results on superinjective simplicial maps of the curve complex, on the automorphism group of the complex of nonseparating curves, and on the Hatcher-Thurston complex of cut systems of curves (some of this work is joint with J. McCarthy and with M. Korkmaz). N. Wahl described a stability theorem for the homology of the mapping class group of non-orientable surfaces which is analogous to Harer’s theorem for orientable surfaces. K. Fujiwara spoke on the geometry of the curve graph showing that the asymptotic dimension of this graph is finite, and that for surfaces of genus \( \geq 2 \) with one boundary component, the dimension is at least two. A description of symplectic structures on Lefschetz fibrations using algebraic properties of the mapping class group was given by M. Korkmaz. At a more algebraic level, N. Kawazumi described recent work on characteristic classes in the mapping class group, in which he constructs higher analogues of the period matrix in order to obtain “canonical” differential forms that represent all the Morita-Mumford classes and their higher relations. R. Cohen gave a talk on joint work with I. Madsen on a generalized Mumford conjecture on the stable cohomology of the mapping class group and a general version of homology stability for that group in the setting of spaces of Riemann surfaces with appropriate boundary conditions in a simply connected target manifold. G. Mondello reported on his work relating the tautological classes to cycles of Witten and Kontsevich, which are constructed combinatorially (using fatgraphs).

3) Quantum theory. R. Kashaev described a new and elegant quantization of a homology bundle over Teichmüller space related to his earlier work. L. Chekhov reported (on his joint work with Penner) quantizing Thurston’s projective lamination space for the once-punctured torus. V. Fock discussed an example from cluster algebras giving an explicit relationship between Teichmüller geometry and representation theory which is related to his recent work with A. Goncharov on higher Teichmüller spaces. Y. Gerber described a new construction of a collection of surface mapping classes with computable quantum invariants leading to new invariants for fibered knots. F. Bonsante gave a report on his recent work with R. Benedetti on constant curvature Lorentzian structures on manifolds that are topologically the product of a hyperbolic surface with the real line.

4) Dynamics. M. Möller reported on joint work with I. Bouw on billiards in relation to Veech surfaces, projective affine groups and Teichmüller curves in the moduli space of curves characterizing these curves by properties of the variation of Hodge structures. The talk by U. Hamenstädt, mentioned in 1) above, involved the Teichmüller geodesic flow on quotient of the space of quadratic differentials by the action of the mapping class group. M. Mirzakhani studied the ergodic properties of natural flows on moduli space in relation to the asymptotic behaviour of simple closed geodesics on hyperbolic surfaces.
4) Complex geometry. Y. Imayoshi gave a talk on joint work with T. Nogi on the complex analytic structure of moduli space together with its Deligne-Mumford compactification. Continuing ideas that originate in work of Kodaira, he described a cut-and-paste construction which produces holomorphic families of closed Riemann surfaces of genus two over a four-punctured torus, to which they associate two holomorphic sections.

5) Higher Teichmüller theory A. Wienhard gave a talk on her recent work with M. Burger and A. Iozzi on representations of the fundamental group of the surface into semisimple Lie groups of Hermitian type. G. McShane described geometric identities for surfaces that are related to Hitchin’s a component of the representation variety of the fundamental group of a compact surface into \( SL(n, \mathbb{R}) \). D. Dumas and S. Kojima spoke on complex projective structures on surfaces, the space \( \mathcal{P}(S) \) of which (equivalence classes) can be considered as a higher-analog of Teichmüller space. \( \mathcal{P}(S) \) is a fibre bundle over Teichmüller space, and like Teichmüller space itself, can be studied from different points of view: complex analysis (via the Schwarzian derivative) and hyperbolic geometry (via Thurston’s grafting map). Some of the most interesting questions in the theory of projective structures relate the two points of view, and the talk by Dumas focused on this relation. Kojima described a geometric parametrization of the moduli space of projective surfaces by cross ratios. He develops (together with S. Mizushima and S. P. Tan) a theory of circle packings in projective geometry which can be traced back to works by Andreev and by Thurston.