Abstract. The subjects of convex and algebraic geometry meet primarily in the theory of toric varieties. Toric geometry is the part of algebraic geometry where all maps are given by monomials in suitable coordinates, and all equations are binomial. The combinatorics of the exponents of monomials and binomials is sufficient to embed the geometry of lattice polytopes in algebraic geometry. Recent developments in toric geometry that were discussed during the workshop include applications to mirror symmetry, motivic integration and hypergeometric systems of PDE’s, as well as deformations of (unions of) toric varieties and relations to tropical geometry.


Introduction by the Organisers

The workshop Convex and Algebraic Geometry was organized by Klaus Altmann (Berlin), Victor Batyrev (Tübingen), and Bernard Teissier (Paris). Both title subjects meet primarily in the theory of toric varieties. These constitute the part of algebraic geometry where all maps are given by monomials in suitable coordinates, and all equations are binomial. The combinatorics of the exponents of monomials and binomials is sufficient to embed the geometry of lattice polytopes in algebraic geometry. Thus, toric geometry and its several generalizations provide a kind of section from polyhedral into algebraic geometry. While this reflects only a thin slice of algebraic geometry, it is general enough to display many important phenomena, techniques, and methods. It serves as a wonderful testing ground for general theories such as the celebrated mirror symmetry in its different flavours. In particular, much of the popularity of toric geometry originates in mathematical physics.
The meeting was attended by almost 50 participants from many European countries, Canada, the USA, and Japan. The program consisted of talks by 23 speakers, among them many young researchers. Most subjects fit more or less into the following main areas:

- **Derived categories, quivers, and (homological) mirror symmetry** (Bondal, Craw, Horja, Maclagan, Perling, Siebert, Ueda)

  One of the major discussions during the meeting concerned the existence of strongly exceptional sequences on toric varieties which consist of line bundles. A full exceptional sequence provides a kind of “basis” for the derived category. While Hille and Perling presented an example that does not carry such a sequence of full length, Bondal suggested a method to link this question to sheaves on the dual real torus that are constructible with respect to a certain stratification.

  In general, one expects to gain exceptional sequences from the universal bundles on moduli spaces. Using this method, Craw constructs those sequences on smooth toric Fano threefolds. In this context, Maclagan and Ueda consider the case of three-dimensional abelian quotient singularities. Ueda investigates the Fukaya category of the corresponding potential on the dual torus explicitly.

  Using mirror symmetry, Horja establishes a connection between the orbifold $K$-theory of toric Deligne-Mumford stacks and solutions to GKZ-hypergeometric $D$-modules.

- **Degenerations and deformations** (Brown, Hausen, Siebert, Süss, Vollmert)

  Gross and Siebert have developed a program to understand mirror symmetry as the duality of certain degeneration data. The special fibers split into toric components, and the degeneration is encoded in a topological manifold $B$ with an affine and a polytopal structure. Duality is now inherited from discrete geometry, and the topology of $B$ reflects the topology of the general fiber. In particular, if $B$ is a ($\mathbb{Q}$-homology) $\mathbb{P}^n$, then this construction might lead to (compact) Hyperkähler varieties.

  Considering, in a special case, a certain contraction of the total space of these families leads to a description of torus actions on algebraic varieties via divisors on their Chow quotients. These divisors carry polytopes or even polyhedral complexes as their coefficients, compare the talks of Hausen, Süss, and Vollmert. In a similar setting, but with an explicit manipulation of Pfaffians, Brown and Reid construct smoothings of certain non-isolated singularities giving rise to four-dimensional flips.

- **Tropical geometry and Welschinger invariants** (Itenberg, Shustin, Siebert)

  The most rigorous degeneration of a variety is the tropical one. Here, everything takes place over the so-called tropical semiring, and one ends up with piecewise linear spaces. In fact, Siebert’s degeneration data mentioned above correspond to these objects.

  Itenberg and Shustin use this approach to calculate the Welschinger invariants, which are a kind of real version of Gromov-Witten invariants. Along the lines of the method of Gathmann and Markwig, there is a recursive formula for theses
invariants. In the case of del Pezzo surfaces, it turns out that both invariants are 
(log-)asymptotically equivalent.

- **Commutative algebra, GKZ-systems, and polytopes** (Bruns, Haase, Hering, Horja, Miller, Pasquier, Stienstra)

  A generalization of toric varieties in a different direction from the torus actions 
mentioned above is given by the notion of spherical varieties. Pasquier considers horospherical Fano varieties and comes up with an adapted notion of (generalized, coloured) reflexive polytopes. Bruns, Haase, and Hering deal with ordinary polytopes and their relations to syzygies of toric varieties.

  For an integral matrix $A$ one obtains a semigroup algebra $\mathbb{C}[NA]$ (leading to the usual affine toric variety) and a GKZ-hypergeometric system of differential equations. The latter depends on a parameter $\beta$, and Miller has reported on a result that relates the set of $\beta$ where the rank of the system jumps to the set of those multidegrees where the semigroup algebra $\mathbb{C}[NA]$ carries local cohomology. In particular, the Cohen-Macaulay property is equivalent to the constant rank condition, answering an old question of Sturmfels.

One of the nighttime discussions gave rise to the suggestion to not include normality in the definition of a toric variety, thus overcoming the cumbersome term of a “not necessarily normal toric variety”.

The workshop was closed on Friday night by an informal piano recital by Benjamin Nill and Milena Hering featuring Strawinsky, Liszt, and Chopin.