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Mini-Workshop: Attraction to Solitary Waves and Related Aspects of Physics

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Abstract. The aim of the miniworkshop is the discussion of the solitary wave asymptotics for nonlinear Hamiltonian partial differential equations and relation to mathematical problems of Quantum Physics. While the existence and orbital stability of solitary waves is fairly well understood, the asymptotic stability of solitary waves is still not understood well. The global attraction of arbitrary solutions of finite energy to the set of solitary waves is not proved but in a few model cases. On the other side, there is now accumulating a great number of recent results that seem to enable us to make a crucial progress in this direction: namely, to prove the solitary asymptotics for the $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation and similar dispersive Hamiltonian systems. On the Quantum Physics side, the workshop contained the thorough discussion of the quantum scattering and renormalization analysis.

Mathematics Subject Classification (2000): 83xx, 35xx.

Introduction by the Organisers

The workshop *Attraction to Solitary Waves and Related Aspects of Physics*, organised by Vladimir Buslaev (St. Petersburg University), Andrew Comech (Texas A&M), Alexander Komech (Universität Wien), and Boris Vainberg (UNC – Charlotte) was held February 10–16, 2008. This meeting was attended with 15 participants with broad geographic representation from Europe and America. This workshop was a blend of researchers with backgrounds in Partial Differential Equations, Harmonic Analysis, and Quantum Field Theory.

The aim of the miniworkshop has been the discussion of current state of the long-time asymptotics for nonlinear Hamiltonian partial differential equations and relation to mathematical problems of Quantum Physics.

The central themes were the orbital and asymptotic stability of solitary waves, quantum scattering, renormalization, and global attraction to solitary waves.

Bohr's transitions as global attraction to solitary waves. According to Bohr's postulates [Boh13], an unperturbed electron runs forever along certain *stationary orbit*, which we denote $|E\rangle$ and call *quantum stationary state*. Once in such a state, the electron has a fixed value of energy E, not losing the energy via emitting radiation. The electron can jump from one quantum stationary state to another,

$$|E_-\rangle \longmapsto |E_+\rangle, \tag{1}$$

emitting or absorbing a quantum of light with the energy equal to the difference of the energies E_+ and E_-. Bohr's second postulate states that the electrons can jump from one quantum stationary state (Bohr's *stationary orbit*) to another.

Bohr's *stationary orbits* were interpreted by Schrödinger as *quasistationary solitary wave solutions* of the form

$$\psi(x,t) = \phi(x)e^{-i\omega t}, \qquad \textit{with} \quad \omega \in \mathbb{R}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{2}$$

We will call such solutions *solitary waves*. Other appropriate names are *nonlinear eigenfunctions* and *quantum stationary states* (the solution (2) is not exactly stationary, but certain observable quantities, such as the charge and current densities, are time-independent indeed). As a consequence, the electron in such a state does not emit the energy and "circles" forever around the nucleus in an atom.

Bohr's *quantum jumps* can be interpreted dynamically as long-time asymptotics

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty, \tag{3}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ generally depend on the trajectory. Then the quantum stationary states should be viewed as the points of the *global attractor* $\mathscr{A}$.

The attraction (3) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{4}$$

that hold for each finite energy solution.

Now let us describe the existing results on solitary waves in the context of dispersive Hamiltonian systems.

Nonlinear wave equations. Well-posedness in the energy space. The nonlinear wave equations take their origin in Quantum Field Theory from the articles by Schiff [Sch51a, Sch51b], who considered the nonlinear Klein–Gordon equation in his research on the classical nonlinear meson theory of nuclear forces. The mathematical analysis of this equation is started by Jörgens [Jör61] and Segal [Seg63a, Seg63b], who studied its global well-posedness in the energy space.

Since then, this equation (alongside with the nonlinear Schrödinger equation) has been the main playground for developing tools to handle more general nonlinear Hamiltonian systems.

Local attraction to zero. The asymptotics of type (4) were discovered first with $\psi_\pm = 0$ in the scattering theory. Segal [Seg66] and then Morawetz and Strauss [Str68, MS72] studied the (nonlinear) scattering for solutions of nonlinear Klein–Gordon equation in $\mathbb{R}^3$. We may interpret these results as *local* (referring to small initial data) attraction to zero:

$$\psi(x,t) \sim \psi_\pm = 0, \qquad t \to \pm\infty. \tag{5}$$

The asymptotics (5) hold on an arbitrary compact set and represent the well-known local energy decay. These results were further extended in [GS79, Kla82, GV85, Hör91].

Solitary waves. Apparently, there could be no *global* attraction to zero (*global* referring to arbitrary initial data) if there are solitary wave solutions of the form $\phi_\omega(x)e^{-i\omega t}$. The existence of solitary wave solutions

$$\psi_\omega(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}^n),$$

with $H^1(\mathbb{R}^n)$ being the Sobolev space, to the nonlinear Klein–Gordon equation (and nonlinear Schrödinger equation) in $\mathbb{R}^n$, in a rather generic situation, was established in [Str77] (a more general result was obtained in [BL83a, BL83b]). Typically, such solutions exist for ω from an interval or a collection of intervals of the real line. We denote the set of all solitary waves by $\mathcal{S}_0$.

While all localized stationary solutions to the nonlinear wave equations in spatial dimensions $n \geq 3$ turn out to be unstable (the result known as "Derrick's theorem" [Der64]), *quasistationary* solitary waves can be orbitally stable. Stability of solitary waves takes its origin from [VK73] and has been extensively studied by Strauss and his school in [GSS87, Sha83, Sha85, SS85].

Local attraction to solitary waves. The asymptotic stability of solitary waves (convergence to a solitary wave for the initial data sufficiently close to it) has been studied by Soffer and Weinstein [SW90, SW92] in the context of nonlinear $\mathbf{U}(1)$-invariant Schrödinger equation with a potential. This theory has been further developed by Buslaev and Perelman [BP93, BP95] and then by others in [PW97, SW99, Cuc01a, Cuc01b, Cuc03, BS03] and other papers. Up to date, there are many open problems. While generically we expect that any orbitaly stable solitary wave is also asymptotically stable, the proof of this statement is only available for just a few cases and under very strong assumptions.

The existing results on orbital and asymptotic stability suggest that the set of orbitally stable solitary waves typically forms a *local attractor*, that is to say, attracts any finite energy solutions that were initially close to it. Moreover, a natural hypothesis is that the set of all solitary waves forms a *global attractor* of all finite energy solutions.

Global attraction to solitary waves. The *global attraction* of type (4) with $\psi_\pm \neq 0$ and $\omega_\pm = 0$ was established in certain models in [Kom91, Kom95, KV96, KSK97, Kom99, KS00] for a number of nonlinear wave problems. There the attractor is the set of all *static* stationary states. Let us mention that this set could be infinite and contain continuous components.

In [Kom03] and [KK07a], the attraction to the set of solitary waves (see Fig. 1) is proved for the Klein–Gordon field coupled to a nonlinear oscillator. In [KK07b], this result has been generalized for the Klein–Gordon field coupled to several oscillators. The paper [KK08] gives the extension to the higher-dimensional setting for a model with the nonlinear self-interaction of the mean field type.

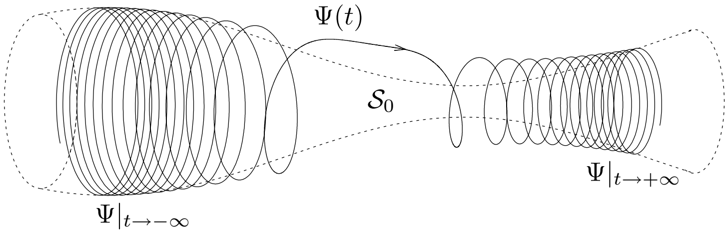

FIGURE 1. For $t \to \pm\infty$, a finite energy solution $\Psi(t)$ approaches (in local energy seminorms) the global attractor $\mathscr{A}$ which coincides with the set of all solitary waves $\mathcal{S}_0$.

Let us mention one more recent advance, [Tao07], in the field of nontrivial (nonzero) global attractors for Hamiltonian PDEs. In that paper, the global attraction for the nonlinear Schrödinger equation in dimensions $n \geq 5$ was considered. The dispersive (outgoing) wave was explicitly specified using the rapid decay of local energy in higher dimensions. The global attractor was proved to be compact, but it was not identified with the set of solitary waves [Tao07, Remark 1.18].

Relation to Quantum Physics. The Quantum Mechanics is formulated in terms of partial differential equations: coupled nonlinear Maxwell-Schrodinger, Maxwell-Dirac, Maxwell-Yang-Mills equations, etc. The Quantum Field Theory is formulated in terms of the correponding second quantized equations. The main goal of our workshop was to achieve the critical concentration of experts in Quantum Theory on one side and in PDEs on another side, to have a thorough discussion of recent advances in both areas and an exchange which could stimulate further progress.

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