The Monge-Ampère equation refers to a fully nonlinear PDE of the form

\[ \det(D^2u(x)) = f, \]

where \( D^2u(x) \) stands for the Hessian of the function \( u \) at \( x \in \Omega \subset \mathbb{R}^n \) and “\( \det \)” denotes the matrix determinant operator. The Monge-Ampère equation is said to be fully nonlinear because its nonlinearity is on the highest-order derivatives of \( u \) appearing in the equation. It is the best known fully nonlinear second-order PDE, and it appears in diverse scientific and engineering fields including antenna design, astrophysics, differential geometry, image processing, nonlinear elasticity, fluid dynamics, and optimal mass transport.

If we let \( \{\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)\} \) denote the eigenvalues of the Hessian \( D^2u(x) \) (which all are real), then by simple matrix algebra facts we have that

\[ \det(D^2u(x)) = \lambda_1(u)\lambda_2(u)\cdots\lambda_n(u) \]

and

\[ \Delta u(x) := \text{tr}(D^2u(x)) = \lambda_1(u) + \lambda_2(u) + \cdots + \lambda_n(u), \]

where “\( \text{tr} \)” denotes the matrix trace operator and \( \Delta \) denotes the Laplace operator. Obviously, the Monge-Ampère operator is nonlinear while the Laplace operator is linear. Indeed, to a large extent the Monge-Ampère equation is to the fully nonlinear (second-order elliptic) equations what the Laplace equation is to the linear (second-order elliptic) equations. In the equation, the right-hand side function \( f \) may or may not depend on the unknown function \( u \). When \( f \) is independent of \( u \), the equation is known as the classical Monge-Ampère equation, otherwise, it is often called a Monge-Ampère-type equation in the literature. The book under review considers both the classical Monge-Ampère equation and Monge-Ampère-type equations such as the prescribed Gauss curvature equation and the optimal mass transport equation, although it mainly focuses on the classical Monge-Ampère equation.

It is a well-known fact that the Monge-Ampère operator \( u \mapsto \det(D^2u) \) is not elliptic in full generality; instead, it only becomes elliptic when restricted on the space of strictly convex functions. This is the main reason why the Monge-Ampère equation is often studied in the space of strictly convex functions. In addition, to study boundary value problems for the Monge-Ampère equation, a boundary condition must be prescribed. Two types of boundary conditions arise from applications for the Monge-Ampère equation; namely, the Dirichlet boundary condition and the so-called second boundary condition given by \( \nabla u(X) \subset Y \) for two given sets \( X, Y \subset \mathbb{R}^n \). The book under review focuses on the Dirichlet boundary value problem for the Monge-Ampère equation, although the second boundary condition is also considered when the optimal mass transport problem is studied.

As stated clearly, the goal of the book is “to give a comprehensive introduction to the existence and regularity theory for the Monge-Ampère equation, and to show...
some selected applications”. This book is a nice complement to an earlier book by C. E. Gutiérrez on the same subject which was published more than fifteen years ago [The Monge-Ampère equation, Progr. Nonlinear Differential Equations Appl., 44, Birkhäuser Boston, Boston, MA, 2001; MR1829162]. In particular, the book under review contains a number of new results which were obtained in the past fifteen years. Moreover, it covers four important applications of the Monge-Ampère equation; namely, the Minkowski problem for curvature measures, Petty’s theorem, the optimal (mass) transport problem with quadratic cost, and the semigeostrophic equations.

This book is based on a series of lectures given by the author at ETH Zürich during the fall of 2014. It is structured following a “historical” path. Chapter 1 gives an introduction to the Monge-Ampère equation and its history. Chapter 2 is devoted to the theory of weak solutions developed by A. D. Aleksandrov in the 1940s. This includes the subdifferential, the Monge-Ampère measure, Aleksandrov’s generalized solutions for the Borel measure-valued source term $f$, Aleksandrov’s maximum principle, the Dirichlet problem and its well-posedness. The chapter ends with a brief discussion of the $C^1$ regularity in 2-D and an application to the Minkowski problem for curvature measures.

Chapter 3 addresses the issue of existence of global smooth solutions. The theory developed in the 1960s and 1980s shows that a smooth solution does exist when the domain and the boundary data are smooth. This is done by the classical continuity method and crucial interior a priori estimates due to A. V. Pogorelov. Chapter 4, which comprises the largest part of the book, studies the interior regularity theory of weak solutions. The main topics include interior $C^{1,\alpha}, W^{2,p}$, and $C^{2,\alpha}$ estimates, and the geometry of solutions mostly studied by L. Caffarelli in the 1990s. All three remaining applications of the Monge-Ampère equation considered in the book are given in this section. Chapter 5 covers some extensions and generalizations of the results described in the previous chapters. These include the Monge-Ampère equation with general right-hand side $f(x, u, \nabla u)$, boundary regularity, singular solutions, the linearized Monge-Ampère equation, and a general class of Monge-Ampère-type equations motivated by optimal mass transport problems and general prescribed Jacobian equations. The book is concluded by an appendix which collects some basic facts from linear algebra, convex geometry, measure theory, nonlinear analysis, and PDEs.

As mentioned by the author in the introduction, the book does not intend to cover all the topics and recent developments in the theory of the Monge-Ampère equation and its variants. Among a few omissions, I would like to point out two which perhaps would have been helpful to the reader. Firstly, the book does not mention the (well-known) viscosity solution concept and its relation to Aleksandrov’s generalized solution concept. This issue was addressed in Gutiérrez’s book. The conclusion is that these two weak solution concepts coincide for the Dirichlet boundary value problem when the domain $\Omega$ is convex and the right-hand side function $f$ is continuous. Secondly, due to their increasing importance to many scientific and engineering fields, numerical approximations of the Monge-Ampère equation and other fully nonlinear PDEs, including Hamilton-Jacobi-Bellman equations, have become important and indispensable. Numerical fully nonlinear PDEs have been thriving and a lot of research has been done and significant progress and advances have been achieved in the last ten years. The reader is referred to two recent survey papers [X. Feng, R. Glowinski and M. Neilan, SIAM Rev. 55 (2013), no. 2, 205–267; MR3049920; M. Neilan, A. J. Salgado and W. Zhang, Acta Numer. 26 (2017), 137–303; MR3653852] for details.

In summary, this new monograph is a very valuable resource concerning the theory of the Monge-Ampère equation. The book is carefully written and well organized. The author did a great job presenting many highly technical and sophisticated analyses in an easy-to-follow manner so the reader will be able to understand and appreciate
contemporary literature on the Monge-Ampère equation. The 125 references give the most relevant sources for further reading. This book provides a concise and accessible introduction to the (existence and regularity) theory of the Monge-Ampère equation. It should be on the shelf of everyone who has an interest in Monge-Ampère-type equations and fully nonlinear second-order PDEs in general. 

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