Degenerate complex Monge-Ampère equations.

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Complex Monge-Ampère equations are one of the most important types of non-linear partial differential equations (PDEs). They arise naturally in differential geometry, namely in establishing the existence of Kähler metrics with certain special properties on compact Kähler manifolds. More specifically: Let \((X, g)\) be a compact Kähler manifold with Kähler form \(\omega\). Eugenio Calabi [in Proceedings of the International Congress of Mathematicians, Amsterdam, 1954. Vol. 2, 206–207, Erven P. Noordhoff N. V., Groningen, 1954; see MR0070535] conjectured that for any real closed \((1,1)\)-form \(\rho'\) on \(X\) such that \[\int_X \rho' = 2\pi c_1(X) \in H^2(X, \mathbb{R}),\] there exists a unique Kähler metric \(g'\) on \(X\) with Kähler form \(\omega'\) such that \([\omega'] = [\omega] \in H^2(X, \mathbb{R})\) and the Ricci form of \(g'\) is the given form \(\rho'\). The Kähler form \(\omega'\) is given by \(\omega + dd^c \varphi\) where \(\varphi\) is the solution of the complex Monge-Ampère equation

\[(\omega + dd^c \varphi)^n = \mu,\]

normalized so that \(\int_X \varphi \omega^n = 0\); here \(\mu = e^h \omega^n\) is a smooth volume form.

Calabi’s conjecture is related to the question of which Kähler manifolds have Kähler-Einstein metrics: A Kähler metric \(g\) is, by definition, Kähler-Einstein if its Ricci form is a multiple at each point of the Kähler form \(\omega\) of the metric \(g\). This question can be reduced to solving the following complex Monge-Ampère equation:

\[(\omega + dd^c \varphi)^n = e^{-\lambda \varphi} \mu,\]

where \(\lambda\) is a real number whose sign is that of the first Chern class of the manifold \(X\).

To solve degenerate complex Monge-Ampère equations, the authors develop different methods:

1. Continuity method. This involves a family of equations

\[\left((\text{CY})_t\right) (\omega + dd^c \varphi_t)^n = |te^h + (1-t)|\omega^n,\]

where \(t \in [0,1]\) and \(\varphi_t\) is a normalized Kähler potential \((\omega + dd^c \varphi_t)\) is a Kähler form such that \(\int_X \varphi_t \omega^n = 0\). These are trivial to solve when \(t = 0\), as \(\varphi_0 = 0\) is the solution. The idea of the continuity method is to show that such an equation can be solved for all \(t \in [0,1]\). Fix \(k \geq 3\) and \(\alpha \in (0, 1)\). Let

\[S = \{ t \in [0,1] : \text{the equation (CY)}_t \text{ has a solution in } PSH(X, \omega) \cap C^{k+2,\alpha}(X) \}\]

Since \(0 \in S\), it suffices to prove that \(S\) is both open and closed in \([0,1]\).

To prove openness, one applies the implicit function theorem to the linearization of the above equation which is invertible. The hardest part is to prove that \(S\) is closed. In order to do this, one finds some hard a priori estimates for the solution \(\varphi_t\) and their higher derivatives: There exists a real number \(\beta > \alpha\) and a uniform constant such that

\[\|\varphi_t\|_{C^{k+2,\alpha}} \leq C\]

for all \(t \in S\). These estimates are established and well explained in Chapters 12 and 14.

The degenerate case is more subtle since when \(\omega\) is merely semi-positive and big, the Laplace operator \(\Delta_\omega\) is no longer invertible. In order to overcome these obstacles, the authors approximate the form \(\omega\) by \(\omega + \varepsilon \omega_X\), where \(\omega_X\) is Kähler and \(\varepsilon \searrow 0\), use non-degenerate solution and pass to the limit.
(2) Variational approach. Consider the following functionals $F_\lambda$:

$$F_\lambda(\varphi) = E(\varphi) + \frac{1}{\lambda} \log \left( \frac{\int_X e^{-\lambda \varphi - h} \omega^n}{V} \right),$$

where $V = \int_X \omega^n = Vol_\omega(X)$ and

$$E(\varphi) = \frac{1}{(n+1)V} \sum_{j=1}^n \int_X \varphi (\omega + dd^c \varphi)^j \wedge \omega^{n-j}.$$ 

Note that the potentials $\varphi$ may exhibit some singularities. It is not even clear that the functionals $F_\lambda$ will make sense.

The complex Monge-Ampère equations

$$\frac{1}{V} (\omega + dd^c \varphi_t)^n = \frac{e^{-\lambda \varphi - h}}{\int_X e^{-\lambda \varphi - h} \omega^n} \omega^n,$$

are the Euler-Lagrange equations of the functionals $F_\lambda$. A potential $\varphi$ is a critical point of $F_\lambda$ if and only if it satisfies the equation (EL). The variational approach consists of finding the extrema of the functionals $F_\lambda$. This approach is well exposed and explained in Chapter 11.

(3) Viscosity approach. The notion of viscosity solution was introduced by Pierre-Louis Lions and Michael G. Crandall to generalize the classical solution to a PDE. The main advantages of this concept are that it allows merely continuous functions to be solutions of some fully nonlinear degenerate PDEs and provides very general existence and uniqueness theorems. The authors of the book adapt the concept for the complex Monge-Ampère equations. In Chapter 6, they give a self-contained presentation of the concept and treat degenerate complex Monge-Ampère on domains of $\mathbb{C}^n$. The concept is developed on the compact Kähler manifold in Chapter 13.

This book is a compilation and expansion of lecture notes of a graduate course given by the authors at Université Paul Sabatier in Toulouse, France. It is divided into four parts and sixteen chapters. It begins with a self-contained presentation of pluripotential theory in domains of $\mathbb{C}^n$. The basic properties of harmonic, subharmonic and plurisubharmonic functions are given in Chapter 1. The authors present the basic theory of positive closed currents in Chapter 2 and develop properties of the complex Monge-Ampère operator and various Dirichlet problems in Chapters 3, 4 and 5. The viscosity approach is used to solve degenerate complex Monge-Ampère equations on domains of $\mathbb{C}^n$ in Chapter 6.

In the second part, the authors transfer and adapt the local pluripotential theory (in domains of $\mathbb{C}^n$) to that of compact Kähler manifolds. They start by reviewing some topics from complex geometry in Chapter 7. By the maximum principle, there are no global plurisubharmonic functions (except constants) on compact manifolds. In this context, the plurisubharmonic functions are replaced by quasip plurisubharmonic ones which are locally given as the sum of a smooth and plurisubharmonic function. In Chapter 8, the authors establish basic properties of these functions. The finite-energy classes are introduced and studied in Chapters 9 and 10.

The third part, which is the central part of the book, is devoted to solving degenerate complex Monge-Ampère equations using different techniques. A variational approach is developed in Chapter 11. Several a priori $L^\infty$ estimates are presented in Chapter 12. The viscosity techniques are adapted to the compact setting in Chapter 13. Higher-order estimates and the smoothness of the solutions to some complex Monge-Ampère equations in the ample locus of $\{\omega\}$ are established in Chapter 14.

The last part gives several applications to differential geometry of the results developed in the previous chapters. In Chapter 15, the authors study the canonical metrics in
Kähler geometry, the Calabi-Yau theorem, Kähler-Einstein metrics and the Riemannian structure on the infinite-dimensional space of Kähler metrics. The existence of singular Kähler-Einstein metrics on mildly singular varieties is treated in Chapter 16.

This book gives a nice presentation of the recent developments of pluripotential theory on compact Kähler manifolds and its applications to Kähler geometry. It is written in comprehensive style, which makes it accessible to advanced students interested in this field and also a very useful reference book for current researchers in complex analysis and differential geometry.

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