Introduction

Spin Geometry is the hidden facet of Riemannian Geometry. It arises from the representation theory of the special orthogonal group SO_n, more precisely, from the spinor representation, a certain representation of the Lie algebra $\mathfrak{so}_n$ which is *not* a representation of SO_n. Spinors can always be constructed locally on a given Riemannian manifold, but globally there are topological obstructions for their existence.

Spin Geometry lies therefore at the cross-road of several subfields of modern Mathematics. Algebra, Geometry, Topology, and Analysis are subtly interwoven in the theory of spinors, both in their definition and in their applications. Spinors have also greatly influenced Theoretical Physics, which is nowadays one of the main driving forces fostering their formidable development. The Noncommutative Geometry of Alain Connes has at its core the Dirac operator on spinors. The same Dirac operator is at the heart of the Atiyah–Singer index formula for elliptic operators on compact manifolds, linking in a spectacular way the topology of a manifold to the space of solutions to elliptic equations. Significantly, the classical Riemann–Roch formula and its generalization by Hirzebruch; the Gauß–Bonnet formula and its extension by Chern; and finally Hirzebruch's topological signature theorem, provide most of the examples in the index formula, but it was the Dirac operator acting on spinors which turned out to be the keystone of the index formula, both in its formulation and in its subsequent developments. The Dirac operator appears to be the primordial example of an elliptic operator, while spinors, although younger than differential forms or tensors, illustrate once again that *aux âmes bien nées, la valeur n'attend point le nombre des années*.

Our book aims to provide a comprehensive introduction to the theory of spinors on oriented Riemannian manifolds, including some (but by no means all) recent developments illustrating their effectiveness. Our primordial interest comes from Riemannian Geometry, and we adopt the point of view that spinors are some sort of "forgotten" tensors, which we study in depth.

Several textbooks related to this subject have been published in the last two or three decades. Without trying to list them all, we mention Lawson and Michelson's foundational *Spin geometry* [LM89], the monograph by Berline, Getzler and Vergne centered on the heat kernel proof of the Atiyah–Singer index theorem [BGV92], and more recently the books by Friedrich [Fri00] and Ginoux [Gin09] devoted to the spectral aspects of the Dirac operator. One basic issue developed in our book which is not present in the aforementioned works is the interplay between spinors and special geometric structures on Riemannian manifolds. This aspect becomes particularly evident in small dimensions $n \leq 8$, where the spin group acts transitively on the unit sphere of the real (half-) spin representation, i.e., all spinors are *pure*. In this way, a non-vanishing (half-) spinor is equivalent to a SU_2-structure for $n = 5$, a SU_3-structure for $n = 6$, a G_2-structure for $n = 7$, and a Spin_7-structure for $n = 8$. Many recent contributions in low-dimensional geometry (e.g., concerning hypo, half-flat, or co-calibrated G_2 structures) are actually avatars of the very same phenomena which have more natural interpretation in spinorial terms.

One further novelty of the present book is the simultaneous treatment of the spin, Spin^c, conformal spin, and conformal Spin^c geometries. We explain in detail the relationship between almost Hermitian and Spin^c structures, which is an essential aspect of the Seiberg–Witten theory, and in the conformal setting, we develop the theory of weighted spinors, as introduced by N. Hitchin and P. Gauduchon, and derive several fundamental identities, e.g., the conformal Schrödinger–Lichnerowicz formula.

The Clifford algebra

We introduce spinors via the standard construction of the Clifford algebra Cl_n of a Euclidean vector space $\mathbb{R}^n$ with its standard positive-definite scalar product. The definition of the Clifford algebra in every dimension is simple to grasp: it is the unital algebra generated by formal products of vectors, with relations implying that vectors anti-commute up to their scalar product:

$$u \cdot v + v \cdot u = -2\langle u, v\rangle.$$

In low dimensions $n = 1$ and $n = 2$, the Clifford algebra is just $\mathbb{C}$, respectively $\mathbb{H}$, the quaternion algebra. Already for $n = 1$, we encounter the extravagant idea of "imaginary numbers", which complete the real numbers and which were accepted only as late as the eighteenth century. Quaternions took another century to be devised, while their generalization to higher-dimensional Clifford algebras is rather evident. The same construction, but with the bilinear form replaced by 0, gives rise to the exterior algebra, while the universal enveloping algebra of a Lie algebra is closely related. The Clifford algebra acts transitively on the exterior algebra, and thus in particular it is nonzero. This algebraic construction has unexpected applications in Topology, as it is directly related to the famous vector field problem on spheres, namely finding the maximal number of everywhere linearly independent vector fields on a sphere.

The spin group

Inside the group of invertible elements of Cl_n we distinguish the subgroup formed by products of an even number of unit vectors, called the *spin group* Spin_n. It is a compact Lie group, simply connected for $n \geq 3$, endowed with a canonical orthogonal action on $\mathbb{R}^n$ defining a non-trivial $2:1$ cover $\mathrm{Spin}_n \to \mathrm{SO}_n$. In other words, Spin_n is the universal cover of SO_n for $n \geq 3$. Every representation of SO_n is of course also a representation space for Spin_n, but there exists a *fundamental representation* of Spin_n which does not come from SO_n, described below. It is this "shadow orthogonal representation" which gives rise to spinor fields.

The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is canonically isomorphic to the matrix algebra $\mathbb{C}(2^{n/2})$ for n even, respectively to the direct sum of two copies of $\mathbb{C}(2^{(n-1)/2})$ for n odd, and thus has a standard irreducible representation on $\mathbb{C}^{2^{[n/2]}}$ for n even and two non-equivalent representations on $\mathbb{C}^{2^{[n/2]}}$ for n odd. In the first case, the restriction to Spin_n of this representation splits as a direct sum of two inequivalent representations of the same dimension (the so-called *half-spin representations*), whereas for n odd the restrictions to Spin_n of the two representations of $\mathbb{C}\mathrm{l}_n$ are equivalent and any of them is referred to as the *spin representation*.

The classification of real Clifford algebras is slightly more involved, and is based on the algebra isomorphisms $\mathrm{Cl}_{n+8} = \mathbb{R}(16) \otimes \mathrm{Cl}_n$. Note that the periodicity of real and complex Clifford algebras is intimately related to the Bott periodicity of real, respectively complex K-theory, which provides a very convenient algebraic setting for the index theorem.

Spinors

The geometric idea of spinors on an n-dimensional Riemannian manifold (M^n, g) is to consider the spin group as the structure group in a principal fibration extending, in a natural sense, the orthonormal frame bundle of M. The existence of such a fibration, called a *spin structure*, is not always guaranteed, since it is equivalent to a topological condition, the vanishing of the second Stiefel–Whitney class $w_2 \in H^2(M, \mathbb{Z}/2\mathbb{Z})$. Typical examples of manifolds which do not admit spin structures are the complex projective spaces of even dimensions. When it is non-empty, the set of spin structures is an affine space modeled on $H^1(M, \mathbb{Z}/2\mathbb{Z})$. For instance, on a compact Riemann surface of genus g, there exist 2^{2g} inequivalent spin structures. In this particular case, a spin structure amounts to a holomorphic square root of the holomorphic tangent bundle $\mathrm{T}^{1,0}M$.

Once we fix a spin structure on M, *spinors* are simply sections of the vector bundle associated to the principal Spin_n-bundle via the fundamental spin representation. In general, we may think of spinors as *square roots of exterior forms*. More to the point, on Kähler manifolds in every dimension, spin structures correspond to holomorphic square roots of the canonical line bundle.

The Dirac operator

In the same way as the Laplacian $\Delta = d^*d$ is naturally associated to every Riemannian metric, on spin manifolds there exists a prominent first-order elliptic differential operator, the Dirac operator. The first instance of this operator appeared indeed in Dirac's work as a *differential* square root of the Laplacian on Minkowski space-time $\mathbb{R}^4$, by allowing the coefficients to be matrices, more precisely the Pauli matrices, which satisfy the Clifford anti-commutation relations. The generalization of Dirac's operator to arbitrary spin manifolds was given by Atiyah–Singer and Lichnerowicz. The spin bundle inherits the Levi-Civita connection from the frame bundle, and hence the spinor bundle is endowed with a natural covariant derivative ∇. Moreover, vector fields act on spinors by Clifford multiplication, denoted by γ. The Dirac operator is then defined as the Levi-Civita connection composed with the Clifford multiplication:

$$\mathcal{D} = \gamma \circ \nabla \colon \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M), \quad \mathcal{D}\Psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \Psi,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame. The Dirac operator satisfies the fundamental Schrödinger–Lichnerowicz formula:

$$\mathcal{D}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{Scal}, \tag{1}$$

where Scal is the scalar curvature function. A spectacular and elementary application of this formula is the Lichnerowicz theorem which says that if the manifold is closed and the scalar curvature is positive, then there is no harmonic spinor, i.e., $\operatorname{Ker}\mathcal{D} = 0$. On the other hand, Atiyah and Singer had computed the index of the Dirac operator on compact even-dimensional spin manifolds. They noted that the spinor bundle on such a manifold is naturally $\mathbb{Z}_2$-graded, and the Dirac operator is odd with respect to this grading. The operator $\mathcal{D}^+$ is defined as the restriction of $\mathcal{D}$ to positive spinors, $\mathcal{D}^+ \colon \Gamma(\Sigma^+ M) \to \Gamma(\Sigma^- M)$, and its index is the Fredholm index, namely the difference of the dimensions of the spaces of solutions for $\mathcal{D}^+$ and its adjoint. The Atiyah–Singer formula gives this index in terms of a characteristic class, the $\widehat{A}$ genus. It follows that M does not admit metrics of positive scalar curvature if $\widehat{A}(M) \neq 0$. It also follows that $\widehat{A}(M)$ is an integer class when M is a spin manifold. The Atiyah–Singer formula for twisted Dirac operators was also a crucial ingredient in the recent classification of inner symmetric spaces and positive quaternion-Kähler manifolds with weakly complex tangent bundle [GMS11].

Another unexpected application of the Dirac operator is Witten's proof of the positive mass theorem for spin manifolds. In this book we present a variant of Witten's approach by Ammann and Humbert, which works on locally conformally flat compact spin manifolds. The idea is to express the mass as the logarithmic term in the diagonal expansion of the Green kernel for the conformal Laplacian, then use a scalar-flat conformal metric to construct a harmonic spinor on the complement of a fixed point, and finally to use the Schrödinger–Lichnerowicz formula (1) and integration by parts to deduce that the mass is non-negative.

Elliptic theory and representation theory

A main focus in this text is on the eigenvalues of the Dirac operator on compact spin manifolds. It appeared desirable to include a self-contained treatment of several (by now, classical) facts about the spectrum of an elliptic differential operator. Once these are established, as a bonus we apply the corresponding results for the Laplacian on functions to prove the Peter–Weyl theorem. Then we develop the representation theory of semisimple compact Lie groups, in order to compute explicitly the spectra of the Dirac operator on certain compact symmetric spaces.

The lowest eigenvalues of the Dirac operator and special spinors

As already mentioned, the Atiyah–Singer index theorem illustrates how the spectrum of the Dirac operator encodes subtle information on the topology and the geometry of the underlying manifold. Another seminal work on the subject is due to N. Hitchin and concerns harmonic spinors; see [Hit74]. He first discovered that, in contrast with the Laplacian on exterior forms, the dimension of the space of harmonic spinors is a conformal invariant which can (dramatically) change with the metric.

A significant part of the book is devoted to a detailed study of the relationship between the spectrum of the Dirac operator and the geometry of closed spin manifolds with positive scalar curvature.

In that context, the Schrödinger–Lichnerowicz formula not only states that on a closed spin manifold with positive scalar curvature there is no harmonic spinor, but also that there is a gap in the spectrum of the square of the Dirac operator. The first important achievement is Friedrich's inequality, which says that the first eigenvalue of the Dirac operator is bounded from below by that of the sphere, the model space of such a family of manifolds. The original proof of Th. Friedrich involved the notion of "modified connection." The inequality may also be proved by using elementary arguments in representation theory, but the simplest proof relies on the Schrödinger–Lichnerowicz formula together with the spinorial Cauchy–Schwarz inequality:

$$|\nabla\Psi|^2 \geq \frac{1}{n}|\mathcal{D}\Psi|^2, \quad \text{for any spinor field } \Psi.$$

Another remarkable consequence of this point of view is that how far this inequality is from being an equality is precisely measured by $|\mathcal{P}\Psi|^2$, where $\mathcal{P}$ is the Penrose operator (also called the Twistor operator).

Closed manifolds for which the first eigenvalue of the Dirac operator (in absolute value) satisfies the limiting case of Friedrich's inequality are called *limiting manifolds*. They are characterized by the existence of a spinor field in the kernel of the Penrose operator (called *twistor-spinor*) which is also an eigenspinor of $\mathcal{D}$. These special spinors are called *real Killing spinors* (this terminology is due to the fact that the associated vector field is Killing).

It has been observed by Hijazi and Lichnerowicz that manifolds having real Killing spinors cannot carry non-trivial parallel forms, hence there is no real Killing spinor on a manifold with "special" holonomy. With this in mind, Friedrich's inequality could be improved in different directions.

First, by relaxing the assumption on the positivity of the scalar curvature. For instance, based on the conformal covariance of the Dirac operator, a property shared with the Yamabe operator (the conformal scalar Laplacian), it is surprising to note that if one considers the Schrödinger–Lichnerowicz formula over a closed spin manifold for a conformal class of metrics, then for a specific choice of the conformal factor, basically a first eigenfunction of the Yamabe operator, it follows that the square of the first eigenvalue of the Dirac operator is, up to a constant, at least the first eigenvalue of the Yamabe operator (this is known as *the Hijazi inequality*). Again the limiting case is characterized by the existence of a real Killing spinor. Another approach is to consider a deformation (which generalizes that introduced by Friedrich) of the spinorial covariant derivative by Clifford multiplication by the symmetric endomorphism of the tangent bundle associated with the energy–momentum tensor corresponding to the eigenspinor. This approach is of special interest in the setup of extrinsic spin geometry.

Secondly, a natural question is to improve Friedrich's inequality for manifolds with "special" holonomy. By the Berger–Simons classification, one knows that among all compact spin manifolds with positive scalar curvature, we have Kähler manifolds and quaternion-Kähler manifolds. They carry a parallel 2-form (the Kähler form) and a parallel 4-form (the Kraines form), respectively. Model spaces of such manifolds are respectively the complex projective space (note that complex projective spaces of even complex dimension are not spin, but are standard examples of Spin^c manifolds) and the quaternionic projective space. It is natural to expect that for these manifolds the first eigenvalue of the Dirac operator is at least equal to that of the corresponding model space. This is actually the case (with the restriction that for Kähler manifolds of even complex dimension, the lower bound turns out to be given by the first eigenvalue of the product of the complex projective space with the 2-dimensional real torus). These lower bounds are due to K.-D. Kirchberg in the Kähler case and to W. Kramer, U. Semmelmann, and G. Weingart in the quaternion-Kähler setup. There are different proofs of these inequalities, but it is now clear that representation theory of the holonomy group plays a central role in the approach. Roughly speaking, the proofs presented here are based on the use of Penrose-type operators given by the decomposition of the spinor bundle under the action, via Clifford multiplication, of the geometric parallel forms characterizing the holonomy.

Thirdly, a natural task is to classify closed spin manifolds M of positive scalar curvature admitting real Killing spinors and characterize limiting manifolds of Kähler or quaternion-Kähler type. For the first family of manifolds, the classification (obtained by C. Bär) is based on the cone construction, i.e., the manifold $\bar{M}$ defined as a warped product of M with the interval $(0, +\infty)$. This warped product is defined in such a way that for $M = \mathbb{S}^n$ then $\bar{M}$ is isometric to $\mathbb{R}^{n+1} \backslash \{0\}$. The cone construction was used by S. Gallot in order to characterize the sphere as being the only limiting manifold for the Laplacian on exterior forms. M. Wang characterized complete simply connected spin manifolds $\bar{M}$ carrying parallel spinors by their possible holonomy groups. Bär has shown that Killing spinors on M are in one-to-one correspondence with parallel spinors on $\bar{M}$, hence he deduced a list of possible holonomies for $\bar{M}$, and consequently a list of possible geometries for M.

The classification of limiting manifolds of Kähler type is due to A. Moroianu. The key idea is to interpret any limiting spinor as a Killing spinor on the unit canonical bundle of the manifold. It turns out that in odd complex dimensions, limiting manifolds of dimension $4l - 1$ are exactly twistor spaces associated to quaternion-Kähler manifolds of positive scalar curvature, and those of dimension $4l + 1$ are the complex projective spaces. In even complex dimensions $m = 2l \geq 4$, the universal cover $\widetilde{M}$ of a limiting manifold M^{2m} is either isometric to the Riemannian product $\mathbb{C}\mathrm{P}^{m-1} \times \mathbb{R}^2$, for l odd, or to the Riemannian product $N^{2m-2} \times \mathbb{R}^2$, for l even, where N is a limiting manifold of odd complex dimension. A spectacular application of these classification results is an alternative to LeBrun's proof of the fact that every contact positive Kähler–Einstein manifold is the twistor space of a positive quaternion-Kähler manifold.

Finally, in the case of compact spin quaternion-Kähler manifolds M^{4m}, it was proved by W. Kramer, U. Semmelmann, and G. Weingart, that the only limiting manifold is the projective space $\mathbb{H}\mathrm{P}^m$. We present here a more elementary proof.

Dirac spectra of model spaces

As pointed out previously, it is important to compute the Dirac spectrum for some concrete examples, and the archetypal examples in geometry are given by symmetric spaces. For these manifolds, computing the spectrum is a purely algebraic problem which can be theoretically solved by classical harmonic analysis methods, involving representation theory of compact Lie groups. Those methods were already well known in case of the Laplacian.

In the last part of the book we give a short self-contained review of the representation theory of compact groups. Our aim is to provide a sort of practical guide in order to understand the techniques involved. From this point of view, representations of the standard examples (unitary groups, orthogonal and spin groups, symplectic groups) are described in detail. In the same spirit, we give an elementary introduction to symmetric spaces. This point of view is also followed to explain the general procedure for an explicit computation of the spectrum of the Dirac operator of compact symmetric spaces, and then the spectrum of the standard examples (spheres, complex and quaternionic projective spaces) is computed in order to illustrate the method.