

# Contents

Introduction . . . . .	1
<b>Part I Basic spinorial material . . . . .</b>	<b>9</b>
<b>1 Algebraic aspects . . . . .</b>	<b>11</b>
1.1 Clifford algebras . . . . .	11
1.1.1 Definitions . . . . .	11
1.1.2 Classification of Clifford algebras . . . . .	18
1.2 Spin groups and their representations . . . . .	20
1.2.1 Spin groups . . . . .	20
1.2.2 Representations of spin groups . . . . .	28
1.2.3 Real and quaternionic structures . . . . .	36
<b>2 Geometrical aspects . . . . .</b>	<b>39</b>
2.1 Spinorial structures . . . . .	39
2.1.1 Spin structures and spinorial metrics . . . . .	39
2.1.2 Spinorial connections and curvatures . . . . .	42
2.2 $\text{Spin}^c$ and conformal structures . . . . .	45
2.2.1 $\text{Spin}^c$ structures . . . . .	45
2.2.2 Weyl structures . . . . .	49
2.2.3 Spin and $\text{Spin}^c$ conformal manifolds . . . . .	52
2.3 Natural operators on spinors . . . . .	56
2.3.1 General algebraic setting . . . . .	57
2.3.2 First-order differential operators . . . . .	59
2.3.3 Basic differential operators on spinor fields . . . . .	61
2.3.4 The Dirac operator: basic properties and examples . . . . .	63
2.3.5 Conformal covariance of the Dirac and Penrose operators . . . . .	69
2.3.6 Conformally covariant powers of the Dirac operator . . . . .	72

2.4	Spinors in classical geometrical contexts . . . . .	73
2.4.1	Restrictions of spinors to hypersurfaces . . . . .	73
2.4.2	Spinors on warped products . . . . .	75
2.4.3	Spinors on Riemannian submersions . . . . .	76
2.5	The Schrödinger–Lichnerowicz formula . . . . .	81
<b>3</b>	<b>Topological aspects . . . . .</b>	<b>85</b>
3.1	Topological aspects of spin structures . . . . .	85
3.1.1	Čech cohomology and principal bundles . . . . .	86
3.1.2	Lifting principal bundles via central extensions . . . . .	88
3.1.3	Stiefel–Whitney classes . . . . .	90
3.2	Topological classification of $\text{Spin}^c$ structures . . . . .	91
3.3	Spin structures in low dimensions . . . . .	93
3.3.1	Dimension 1 . . . . .	93
3.3.2	Dimension 2 . . . . .	94
3.3.3	Dimensions 3 and 4 . . . . .	95
3.4	Examples of obstructed manifolds . . . . .	96
<b>4</b>	<b>Analytical aspects . . . . .</b>	<b>99</b>
4.1	Fourier transform . . . . .	99
4.2	Pseudo-differential calculus . . . . .	101
4.2.1	Symbols . . . . .	101
4.2.2	Asymptotic summation . . . . .	102
4.3	Pseudo-differential operators . . . . .	104
4.4	Composition of pseudo-differential operators . . . . .	107
4.5	Action of diffeomorphisms on pseudo-differential operators . . . . .	108
4.6	Pseudo-differential operators on vector bundles . . . . .	109
4.7	Elliptic operators . . . . .	112
4.8	Adjoint . . . . .	113
4.9	Sobolev spaces . . . . .	113
4.10	Compact operators . . . . .	118
4.11	Eigenvalues of self-adjoint elliptic operators . . . . .	121
<b>Part II</b>	<b>Lowest eigenvalues of the Dirac operator on closed spin manifolds . . . . .</b>	<b>125</b>
<b>5</b>	<b>Lower eigenvalue bounds on Riemannian closed spin manifolds . . . . .</b>	<b>127</b>
5.1	The Lichnerowicz theorem . . . . .	127
5.2	The Friedrich inequality . . . . .	129
5.3	Special spinor fields . . . . .	133
5.4	The Hijazi inequality . . . . .	137
5.5	The action of harmonic forms on Killing spinors . . . . .	140

5.6	Other estimates of the Dirac spectrum . . . . .	142
5.6.1	Moroianu–Ornea’s estimate . . . . .	142
5.7	Further developments . . . . .	146
5.7.1	The energy–momentum tensor . . . . .	147
5.7.2	Witten’s proof of the positive mass theorem and applications . . . . .	148
5.7.3	Further applications . . . . .	158
<b>6</b>	<b>Lower eigenvalue bounds on Kähler manifolds . . . . .</b>	<b>161</b>
6.1	Kählerian spinor bundle decomposition . . . . .	161
6.2	The canonical line bundle . . . . .	164
6.3	Kählerian twistor operators . . . . .	166
6.4	Proof of Kirchberg’s inequalities . . . . .	170
6.5	The limiting case . . . . .	172
<b>7</b>	<b>Lower eigenvalue bounds on quaternion-Kähler manifolds . . . . .</b>	<b>175</b>
7.1	The geometry of quaternion-Kähler manifolds . . . . .	176
7.2	Quaternion-Kähler spinor bundle decomposition . . . . .	179
7.3	The main estimate . . . . .	182
7.4	The limiting case . . . . .	190
7.5	A systematic approach . . . . .	202
7.5.1	General Weitzenböck formulas . . . . .	202
7.5.2	Application to quaternion-Kähler manifolds . . . . .	206
7.5.3	Proof of the estimate . . . . .	210
<b>Part III</b>	<b>Special spinor field and geometries . . . . .</b>	<b>213</b>
<b>8</b>	<b>Special spinors on Riemannian manifolds . . . . .</b>	<b>215</b>
8.1	Parallel spinors on spin and $\text{Spin}^c$ manifolds . . . . .	215
8.1.1	Parallel spinors on spin manifolds . . . . .	215
8.1.2	Parallel spinors on $\text{Spin}^c$ manifolds . . . . .	219
8.2	Special holonomies and relations to warped products . . . . .	223
8.2.1	Sasakian structures . . . . .	223
8.2.2	3-Sasakian structures . . . . .	225
8.2.3	The exceptional group $G_2$ . . . . .	227
8.2.4	Nearly Kähler manifolds . . . . .	229
8.2.5	The group $\text{Spin}_7$ . . . . .	233
8.3	Classification of manifolds admitting real Killing spinors . . . . .	236
8.4	Detecting model spaces by Killing spinors . . . . .	239
8.5	Generalized Killing spinors . . . . .	241
8.6	The Cauchy problem for Einstein metrics . . . . .	243

<b>9</b>	<b>Special spinors on conformal manifolds</b>	251
9.1	The conformal Schrödinger–Lichnerowicz formula	251
9.2	Parallel spinors with respect to Weyl structures	254
9.2.1	Parallel conformal spinors on Riemann surfaces	254
9.2.2	The non-compact case	256
9.2.3	The compact case	262
9.3	A conformal proof of the Hijazi inequality	263
<b>10</b>	<b>Special spinors on Kähler manifolds</b>	265
10.1	An introduction to the twistor correspondence	265
10.1.1	Quaternion-Kähler manifolds	265
10.1.2	3-Sasakian structures	267
10.1.3	The twistor space of a quaternion-Kähler manifold	268
10.2	Kählerian Killing spinors	269
10.3	Complex contact structures on positive Kähler–Einstein manifolds	272
10.4	The limiting case of Kirchberg’s inequalities	275
<b>11</b>	<b>Special spinors on quaternion-Kähler manifolds</b>	279
11.1	The canonical 3-Sasakian $SO_3$ -principal bundle	280
11.2	The Dirac operator acting on projectable spinors	283
11.3	Characterization of the limiting case	288
11.4	Conclusion	299
<b>Part IV</b>	<b>Dirac spectra of model spaces</b>	301
<b>12</b>	<b>A brief survey on representation theory of compact groups</b>	305
12.1	Reduction of the problem to the study of irreducible representations	305
12.2	Reduction of the problem to the study of irreducible representations of a maximal torus	315
12.3	Characterization of irreducible representations by means of dominant weights	321
12.3.1	Restriction to semi-simple simply connected groups	321
12.3.2	Roots and their properties	322
12.3.3	Irreducible representations of the group $SU_2$	325
12.3.4	Proof of the fundamental properties of roots	327
12.3.5	Dominant weights	334
12.3.6	The Weyl formulas	339

12.4	Application: irreducible representations of the classical groups $SU_n$ , $Spin_n$ , and $Sp_n$ . . . . .	343
12.4.1	Irreducible representations of the groups $SU_n$ and $U_n, n \geq 3$ . . . . .	343
12.4.2	Irreducible representations of the groups $Spin_n$ and $SO_n, n \geq 3$ . . . . .	348
12.4.3	Irreducible representations of the group $Sp_n$ . . . . .	361
<b>13</b>	<b>Symmetric space structure of model spaces</b> . . . . .	<b>369</b>
13.1	Symmetric space structure of spheres . . . . .	372
13.2	Symmetric space structure of the complex projective space . . . . .	374
13.3	Symmetric space structure of the quaternionic projective space . . . . .	375
<b>14</b>	<b>Riemannian geometry of model spaces</b> . . . . .	<b>379</b>
14.1	The Levi-Civita connection . . . . .	381
14.2	Spin structures . . . . .	388
14.3	Spinor bundles on symmetric spaces . . . . .	390
14.4	The Dirac operator on symmetric spaces . . . . .	392
<b>15</b>	<b>Explicit computations of the Dirac spectrum</b> . . . . .	<b>395</b>
15.1	The general procedure . . . . .	395
15.2	Spectrum of the Dirac operator on spheres . . . . .	407
15.3	Spectrum of the Dirac operator on the complex projective space . . . . .	409
15.4	Spectrum of the Dirac operator on the quaternionic projective space . . . . .	412
15.5	Other examples of spectra . . . . .	414
	 Bibliography . . . . .	 421
	 Index . . . . .	 451