Introduction

Valuation theory has developed in many directions since the first International Valuation Theory Conference was held in Saskatoon in 1999. The contributions to this volume present a sampling of this progress, in classical subjects such as the theory of valued fields, ramification theory (in particular the defect), anabelian geometry and local uniformization, as well as in subjects of more recent interest such as tame (or definable) geometry, dynamical systems over valued fields and the geometry of spaces of valuations (or of places) and the analysis which one can do on these spaces in spite of the fact that they are far from being manifolds. There are also papers dealing with the valuation theory of Noetherian rings, for example the problem of describing all extensions of a valuation on a Noetherian excellent local ring to its completion, the characterization of the semigroups consisting of the values taken by a valuation on a Noetherian local domain, the comparison of valuations (Izumi's theorem) and the dicritical divisors associated to a rational function, or a pencil of curves, on a surface, which appear in particular in some approaches to the Jacobian problem. Finally, papers which are close in spirit to the valuation theoretic approach to singularity theory are included in this volume. We now go into more detail.

A question of fundamental importance is the existence of local-global principles, both of cohomological and/or arithmetical nature. Recently it was shown by Harbater, Hartmann, Krashen, and further, Colliot-Thélène, Parimala, Suresh, that the quadratic forms over function fields of curves over complete discrete valuation rings satisfy arithmetical local-global principles for (an)isotropy, and fundamental results were proven about cohomological local-global principles over such fields.

On the other hand, even in the case of global fields there are no local-global principles for the existence of smooth points on varieties. But it was shown that for a global field K and any finite set S of places of K, the maximal totally split extension K^S above S satisfies a local-global principle for the existence of smooth points on varieties (by results of Roquette, Moret-Bailly, Green–Pop–Roquette, Geyer–Jarden, and others). This fact was used by Pop to completely describe the absolute Galois group of K^S using valuation theoretical methods. Nevertheless, for arbitrary fields K endowed with finite sets S of arbitrary valuations, little is known about K^S. In their article in this volume, Bary-Soroker and Fehm prove that K^S is not a Hilbertian field, provided that S is a finite set of discrete valuations. In an appendix, Pop generalizes this to finite sets S of arbitrary valuations. But it remains a central open question whether K^S satisfies a local-global principle as global fields K do.

On the constructive side of things, valuation theoretical methods were used to obtain Galois extensions with given properties, e.g., to realize finite groups over rational function fields $K(t)$ for K a large field, and even to solve finite split embedding problems over such fields. Because of the growing importance of large fields in other areas of research (e.g., they show up in a characterization of a class of extremal fields by Azgin–Kuhlmann–Pop), a comprehensive survey on large fields by Pop is included in this volume.

Valuation theory is a key tool in anabelian birational geometry, both in Grothendieck's arithmetical context as well as in Bogomolov's geometrical context. The problem here is to recover the arithmetically and/or geometrically significant valuations using the given Galois theoretical information. In his paper in this volume, Topaz does this in greater generality, building on earlier results by Ware, Koenigsmann, Efrat, and Bogomolov–Tschinkel. In his approach he combines previous ideas, both that of flag functions of Bogomolov–Tschinkel and that of rigid elements combined with Milnor K-Theory of Koenigsmann, Efrat and others. Yet another essential problem here is the question about the first order definability of Henselian valuations. This problem had been studied intensively by Koenigsmann, and Jahnke–Koenigsmann.

The survey paper of Xiao and Zhukov is published in the journal *Algebra i Analiz*, with English translation in the *St. Petersburg Mathematical Journal*. But as it originally grew out of our conference and presents an excellent survey on the ramification theory of complete discrete value fields with arbitrary residue fields, the editors decided to include it in this volume as a reprint; they would like to thank the publishers of *Algebra i Analiz*, of the *St. Petersburg Mathematical Journal*, and of this volume for agreeing to this arrangement.

A central theme in valuation theory and its applications is the description of valuations, or more particularly, of their extensions. Herrera Govantes and his coauthors survey the known results about the classification of extensions of a valuation on an excellent local domain to its completion.

Connected with the problem of description and classification is also the question which structures can be found, or defined, on various spaces of valuations, and which topological spaces appear as such spaces. One can ask the same question for spaces of orderings, where it has been successfully studied by several authors. In comparison, even for the spaces of valuations that are compatible with orderings, or equivalently, for spaces of real places, much less appears to be known and some obvious questions have remained open till the present day. Gondard provides a survey on what is presently known about spaces of $\mathbb{R}$-places. These are places whose residue fields are archimedean ordered and hence can be seen as ordered subfields of $\mathbb{R}$.

Nonstandard models of the reals are nonarchimedean ordered and possess a canonical $\mathbb{R}$-place, whose associated valuation is called the natural valuation. A useful description of such a valuation is obtained when the valued field is represented as a subfield of a power series field (also called "Hahn field") with its canonical valuation. In particular, the study of the reals with exponentiation by van den Dries, Macintyre and Marker and other authors used truncation closed embeddings of their nonstandard models in power series fields. This means that the truncation of every power series in the image lies again in the image. In his paper in this volume, van den Dries studies which operations on subgroups, subrings and subfields preserve the property of being truncation closed. Matusinski gives a survey on the use of power series fields in the study of the asymptotic behaviour of functions on the reals, which is encoded in so-called "Hardy fields". In particular, he discusses the possible differential structures on power series fields. Kaplansky had shown that valued fields which have the same characteristic as their residue field can be embedded in suitable power series fields, but it is an open question whether the same can be done preserving a given differential structure on them.

One of the classical tools for the description of extensions of valuations is the notion of key polynomials, which was used by Ostrowski, formally introduced by MacLane and further developed by Vaquié. In their book "The Valuative Tree", Favre and Jonsson used key polynomials to construct a tree which describes the set of all normalized valuations centered at $\mathbb{C}[[x,y]]$. Granja generalized this result to the set of normalized valuations centered at any two-dimensional regular local domain. The paper of Granja, Martínez and Rodríguez in this volume presents a non-commutative extension to the case of a skew polynomial ring over a division ring D.

Novacoski closes a gap that appeared in the mentioned works of Favre-Jonsson and of Granja and discusses the various topologies on the trees with which they work.

Interestingly, trees can also be used to describe the structure induced by a single valuation. This is exploited in the notion of c-minimality, which is an analogue for valuations of the very important notion of o-minimality. Cubides Kovacsics introduces the reader to c-minimality and its main results (such as a cell decomposition theorem by Haskell and Macpherson), often with simplified proofs.

Another tool for the description and construction of extensions of valuations is the notion of pseudo Cauchy sequences, developed by Ostrowski and Kaplansky. Blaszczok uses them to construct valuations that are scary for algebraic geometers and model theorists alike as they allow infinite towers of finite extensions with nontrivial defect.

By the work of Kuhlmann it has become well known that the defect is a main obstruction in local uniformization and the model theory of valued fields in positive characteristic. Kuhlmann and Naseem undertake a close study of the defect that appears in valued function fields. The defect can not only be defined for finite extensions, but also for valuations on algebraic function fields that satisfy equality in the Abhyankar inequality (if such a valuation is trivial on the ground field, then it is called "Abhyankar valuation" and has no defect at all). Further, the defect is divided into a part that vanishes when passing to the completion, and the complementary part that remains.

A vast body of literature exists which deals with the extensions of a valuation from a field K to the rational function field $K(X)$. Those for which also the residue field extension is transcendental are of particular interest for applications, and they also appear in connection with key polynomials. They are a special case of Abhyankar valuations. Distinguished pairs are a suitable tool for the description of such extensions. In their survey paper, Aghigh, Bishnoi, Kumar and Khanduja describe how they can be used to prove irreducibility criteria for polynomials over K and how they relate to valued algebraic extensions of K; we note that here again, the defect plays a considerable role.

It was observed by several authors that in the case of valued fields of positive characteristic, the defect is intimately connected with additive polynomials. Moreover, van den Dries and Kuhlmann have shown that the images of additive polynomials in several variables over Laurent series fields over finite fields have an important elementary property, called the Optimal Approximation Property. In his paper in this volume, Durhan shows the same over perfect valued fields of positive characteristic for which no algebraic extension has a nontrivial defect.

Kuhlmann had shown that algebraic function fields with Abhyankar valuations that are trivial on the ground field do not admit algebraic extensions with nontrivial defect. Based on this fact, Knaf and Kuhlmann proved local uniformization for all Abhyankar valuations. In his paper in this volume, Teissier extends this result and recent work of Temkin, proving local uniformization for Abhyankar valuations of excellent equicharacteristic local domains with an algebraically closed residue field. As a byproduct, he obtains an alternative proof for the absence of the defect.

At the conference, Ram Abhyankar asked for a proof of the fact that local uniformization for valuations of arbitrary (real) rank can be reduced to the rank one case. This was commonly assumed to be "folklore", but no proof in the literature was known. In their paper in this volume, Novacoski and Spivakovsky provide a detailed proof, for various versions of local uniformization.

Boucksom, Favre and Jonsson reinterpret and strengthen a classical theorem of Izumi on the comparison of divisorial valuations centered at a closed point of a normal algebraic variety Y over an algebraically closed field as the Lipschitz continuity of certain functions on the dual simplicial complex of the simple normal crossing exceptional divisor of a projective birational map $\pi\colon X \to Y$ with X non-singular.

Cassou-Noguès and Libgober relate Hodge-theoretical invariants of local systems on the complement in a small ball $\mathbf{B}$ in the complex plane of a complex analytic plane curve singularity to the "Newton tree" of the singularity, which encodes a toroidal resolution of the curve, where $\mathbf{B}$ is transformed into a toroidal variety and the curve singularity is resolved.

Cossart, Piltant and Reguera undertake an in-depth study of the graded algebras associated to divisorial valuations on the ring of a rational surface singularity and relate their structure to the dual graph and other invariants of the surface singularity.

Cossart, Matusinski and Moreno revisit the notion, algebraized by Abhyankar and Luengo, of a dicritical divisor of a pencil of plane curves in the affine plane, which is closely related to Rees valuations and appears repeatedly in attacks on the Jacobian problem.

Cutkosky provides a survey of what is known about the semigroups of values which Krull valuations can take on Noetherian local domains, and gives a complete description in the case of regular two dimensional local rings.

Halupczok shows how one can define stratifications for algebraic varieties (in fact definable sets) over a Henselian valued field and relates these stratifications to the usual Whitney stratifications, throwing a new light on them.

Hrushovski provides an exposition of a model-theoretic framework for algebraic (or rather definable) geometry over valued fields, discussing definable types and the classification of imaginaries, and establishes a connection with Berkovich analytic geometry.

Mourtada studies in detail the jet schemes of some rational double point singularities and shows the existence of a correspondence between certain irreducible components of some jet schemes and exceptional divisors appearing in the embedded resolution of the singularity, thus suggesting that there might be an embedded version of the Nash correspondence.

Yurova studies the ergodicity properties of 1-Lipschitz transformations on the 2-adic sphere.

Finally, valuation theory is not only about valuations of fields, rings and abelian groups and the underlying ultrametric, but it also considers generalizations of the notion of "valuation", and alternate concepts. Relaxing two of the three axioms for valuations leads to quasi-valuations. Sarussi studies the topologies they induce and proves a version of the approximation theorem for them.