This book is based on lectures given by the authors, and in particular on a postgraduate course taught by W. Schlag in 2010 at ETH of Zürich. It concerns dispersive, Hamiltonian partial differential equations (namely Klein-Gordon, wave and Schrödinger equations) with a focusing nonlinearity. The linearity is also assumed to be subcritical or critical with respect to the Hamiltonian of the equation, the energy $E$. The main purpose of the book is to expose recent results of the authors on the global dynamics of the equation.

In the case of a defocusing nonlinearity, i.e. when the sign of the nonlinearity is such that all terms in the energy are positive, the dynamics is well known. Any finite-energy solution is global and (if the power of the nonlinearity is not too small) scatters to a linear solution as $t$ goes to infinity.

The book under review concerns only the focusing case, for which the dynamics is much more complicated. In this case, blow-up in finite time may occur. There also exist stationary or time-periodic solutions, among which the ground state (the one with minimum energy), usually denoted by $Q$, plays a particular role. Previous works have shown that initial data with energy strictly below the energy of $Q$ split into two open sets, characterized by the sign of a particular functional, leading either to finite-time blow-up in both time directions, or to a global solution. Furthermore, using the compactness/rigidity method initiated by C. E. Kenig and F. Merle [see Invent. Math. 166 (2006), no. 3, 645–675; MR2257393], one can prove that the global solutions in this regime scatter in both time directions.

The main topic of this book is the dynamics of solutions with energy slightly above the ground state energy. It has been shown by the authors, sometimes in collaboration with Joachim Krieger, that for many focusing dispersive equations, the set of initial data with energy below $E(Q) + \varepsilon$ (for a small, positive $\varepsilon$) splits into three sets according to the behaviour of the solution for positive times. Namely, the sets of initial data such that the corresponding solutions respectively blow up in finite time or scatter to linear solutions are open. The complementary of these two sets is a codimension 1 manifold (the center-stable manifold near $Q$) of the set of initial data. A solution with initial data in this manifold is global for positive times and asymptotically decouples into a sum of the ground state (with a possible modulation) and a solution of the linear equation. Looking similarly at the asymptotic behaviour for negative times, one can prove that all nine combinations of dynamics are possible.

Compared to the proof of the dichotomy of the dynamics below the ground state energy, the proof of this “nine sets theorem” needs two new ingredients, the construction of the center stable manifold near $Q$ and a one-pass theorem that excludes almost-homoclinic orbits (solutions starting in a neighborhood of $Q$, leaving this neighborhood and coming back close to $Q$, possibly after a long time).

The book under review considers, as a typical example, the cubic, focusing, Klein-Gordon equation in space-dimension 3,

\begin{equation}
\partial_t^2 u - \Delta u + u = u^3, \quad t \in I \subset \mathbb{R}, \quad x \in \mathbb{R}^3,
\end{equation}
with initial data \((u, \partial_t u)(0)\) in the energy space \((H^1 \times L^2)(\mathbb{R}^3)\). In most of the book, radial symmetry is also assumed.

The book gives a complete, self-contained proof of the “nine sets theorem” for radial solutions of (1). The proof of the analogous theorem for other dispersive equations (sketched in some cases in Chapter 6) relies on very similar ideas, and the choice of the radial Klein-Gordon equation, which admits finite speed of propagation and excludes the scaling and translation invariances, is made to avoid additional technical difficulties.

This is an excellent book, which also gives a very complete introduction to the modern theory of focusing nonlinear dispersive partial differential equations. Apart from the classical Strichartz estimates and variational techniques, and the more recent compactness/rigidity argument (all exposed in Chapter 2), the book emphasizes a point of view inherited from the theory of dynamical systems which is quite standard in the case of dissipative equations, but maybe more original in the setting of dispersive equations (see for example [J. M. Ball, in Nonlinear elasticity, 93–160, Academic Press, New York, 1973; see MR0324993] and [P. W. Bates and C. K. R. T. Jones, in Dynamics reported, Vol. 2, 1–38, Dynam. Report. Ser. Dynam. Systems Appl., 2, Wiley, Chichester, 1989; MR1000974] for pioneering works on the subject). This book will become a reference for anyone interested in this point of view, which will certainly play an important role in further developments in the field.

Chapter 1 is an introductory chapter, which includes a summary of the book, as well as historical and numerical aspects that are not treated in the following chapters.

Chapter 2 gives preliminaries on equation (1): Strichartz estimates, local well-posedness, existence of the ground state and (following [L. E. Payne and D. H. Sattinger, Israel J. Math. 22 (1975), no. 3-4, 273–303; MR0402291; S. Ibrahim, N. Masmoudi and K. Nakanishi, Anal. PDE 4 (2011), no. 3, 405–460; MR2872122]) dichotomy finite-time blow-up/scattering below the ground state energy. This chapter is a very clear introduction to the field.

Chapter 3 concerns solutions that stay close to the ground state \(Q\). The center-stable manifold near \(Q\) is constructed, using two different methods (due respectively to Bates and Jones [op. cit.] and to Lyapunov and Perron). A key point is the existence of a negative eigenvalue for the linearized operator around \(Q\), yielding a behaviour which is typical of hyperbolic dynamics. Dispersive estimates for the linearized evolution, needed in the approach of Lyapunov and Perron, are also proven. In Section 3.5, the authors explain the more intricate construction of the center-stable manifold for the cubic NLS equation, following [M. Beceanu, Comm. Pure Appl. Math. 65 (2012), no. 4, 431–507; MR2877342].

Chapter 4 concerns solutions of (1) with energy slightly above the energy of \(Q\) which do not stay asymptotically close to \(Q\). The key result of this chapter is the one-pass theorem, whose proof uses the linearized evolution near \(Q\) (describing the ejection process from \(Q\)), and a crucial virial type argument.

Chapter 5 closes the proof of the “nine sets theorem”, combining the results of the two previous chapters with the compactness/rigidity method to treat the case of scattering solutions.

Chapter 6 presents similar results in other contexts: nonradial solutions for (1), one-dimensional Klein-Gordon equations, the cubic radial nonlinear Schrödinger equation in three space dimensions and energy-critical wave equations. In each case, ideas of proofs are given, highlighting the new difficulties with respect to equation (1).

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