Mathematical foundations of supersymmetry.
EMS Series of Lectures in Mathematics.

The mathematical theory of supermanifolds was introduced in the 1970s, notably by F. A. Berezin, to provide a mathematically well-defined receptacle for the quantum field theory of bosonic and fermionic fields. Through recent advances in the theory of Lie superalgebra representations, and numerous applications in physics, from string theory to condensed matter, it has gained renewed attention.


The text under review intends to partially ameliorate this somewhat unsatisfactory state. It is a rigorous and formal introduction to the basic theory of (real differentiable) supermanifolds and Lie supergroups. In addition, it contains two chapters introducing
the foundations of supervarieties and (affine) algebraic supergroups (over the complex numbers). In that, it restricts itself to the commendable but modest aim of providing a reference for the super-beginner, with a view towards applications in Lie theory. Thus, the reader will not find a discussion of advanced geometric topics such as bundles with or without $G$-structure, proper actions and orbit types, super-Riemannian or supersymplectic manifolds; nor does the book contain an account of the notoriously difficult integration theory on supermanifolds. Also curiously, despite their importance, complex supermanifolds are introduced, but not thoroughly discussed.

That said, however, the volume is a slim but solid companion that anyone venturing into previously unfamiliar super-territory will be happy to own. In particular, the discussion of orbits (through ordinary points) of supergroup actions is a definite improvement on previous treatments of this topic in the literature.

Let us end with a brief commentary and table of contents. After basics on super-linear algebra (Chapter 1), sheaves and the functor of points are introduced in Chapter 2. The latter point of view is emphasised throughout the book, which is an essential plus of the text—this is the way it should be done. The approach is illustrated by first discussing superschemes (in Chapters 2 and 3), and then real smooth supermanifolds (in Chapter 4). Contrary to Leĭtes’s text, the global objects are introduced more quickly, and vector fields are defined correctly as sheaf derivations, rather than algebra derivations (so that the definition goes through without the assumption that the structure sheaf be $c$-soft). Chapter 5 concerns local normal forms of morphisms, where besides standard material on the inverse function theorem, morphisms of constant rank are considered, later to be applied to orbit morphisms. In Chapter 6, the Frobenius Theorem for distributions is obtained. This is later applied to the construction of homogeneous superspaces. In Chapters 7–9, Lie supergroups and their actions are considered. In detail, in Chapter 7, Kostant’s correspondence of Harish-Chandra superpairs and Lie supergroups is discussed, using Koszul’s construction. It is nice to see this presented in textbook form. Chapter 8 concerns actions, including a definition of the stabiliser group and the orbit morphism. In Chapter 9, homogeneous superspaces are introduced using the Frobenius theorem. This is complemented by a valuable discussion of the $T$-valued points of such a homogeneous superspace. Finally, Chapters 10 and 11 concern superschemes and algebraic supergroups over the complex numbers. The book is rounded off by three appendices, notably one on Lie superalgebras and their representations (written with the assistance of I. Dimitrov).

In summary, this is a nice textbook that any advisor may sincerely recommend to students entering research in supergeometry or Lie supergroups.

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